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Fibration structures in toric Calabi-Yau Fourfolds

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Abstract

In the context of string dualities, fibration structures of Calabi-Yau manifolds play a prominent role. In particular, elliptic and K3 fibered Calabi-Yau fourfolds are important for dualities between string compactifications with four flat space-time dimensions. A natural framework for studying explicit examples of such fibrations is given by Calabi-Yau hypersurfaces in toric varieties, because this class of varieties is sufficiently large to provide examples with very different features while still allowing a large degree of explicit control. In this paper, many examples for elliptic K3 fibered Calabi-Yau fourfolds are found (not constructed) by searching for reflexive subpolyhedra of reflexive polyhedra corresponding to hypersurfaces in weighted projective spaces. Subpolyhedra not always give rise to fibrations and the obstructions are studied. In addition, perturbative gauge algebras for dual heterotic string theories are determined. In order to do so, all elliptically fibered toric K3 surfaces are determined. Then, the corresponding gauge algebras are calculated without specialization to particular polyhedra for the elliptic fibers. Finally, the perturbative gauge algebras for the fourfold fibrations are extracted from the generic fibers and monodromy.

1 Introduction

Traditionally, there used to be several different string theories, all of which had some advantages and some drawbacks. Though by construction the most unnatural of the string theories, the heterotic string theories $Het_{E8\times E_8}$ and $Het_{SO(32)}$ appeared as the best candidates for providing a realistic theory for our world due to the very direct approach of incorporating a gauge group. This gauge group can then be broken to the gauge group $SU(3) \times SU(2) \times U(1)$ of the standard model in the process of compactification to four flat spacetime dimensions. Geometric compactifications of heterotic string theories are defined by a Calabi-Yau manifold and a stable vector bundle on it[2]. The purely super-

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[2]The conditions for these bundles are most easily solved by embedding the vector bundle connection in the tangent bundle's connection. For a long time, these were the only known solutions. They share the common drawback of producing too high a degree of supersymmetry, namely $N = (2, 2)$ worldsheet supersymmetry leading to $N = 2$ space-time supersymmetry. The phenomenologically desired $N = 1$ space-time supersymmetry is obtained by considering more general bundles leading to $N = (0, 2)$ worldsheet supersymmetry.

symmetric closed string theories of type IIA and IIB are much more naturally defined[3] and compactifications do not involve the complications of specifying a vector bundle in addition to the Calabi-Yau compactification space. Apart from too high an amount of supersymmetry, their biggest disadvantage is their lack of nonabelian gauge groups. In fact, this is only true at first sight. Nonabelian gauge groups *can* be included by allowing the compactification spaces to have singularities.

In recent years, the picture has changed by the discovery of a host of dualities connecting the different types of superstring theories with each other and also with some new theories. Together with singular transitions between different compactifications this opens the dreamer's perspective of dealing with just a single theory, which by some yet unknown dynamics forces the universe into its observed state.

The simplest examples of such dualities are T-duality and (not unrelated) mirror duality. These dualities connect different types of string theories, namely the type IIA and IIB superstring theories in their perturbative regime. More general dualities are not as simple as these and usually even connect one theory in its perturbative regime with another one in the limit of strong coupling. Hence, they are only visible when including solitonic degrees of freedom, namely branes and in particular D-branes.

In addition to dualities between different string theories, dualities with 11-dimensional supergravity were discovered. This led to the conjecture of an 11-dimensional quantum theory, from which all the different string theories and 11-dimensional supergravity can be obtained as different limits. For this theory, called M-theory, a perturbative description with membranes as fundamental objects was proposed[4].

Yet another player in the duality web is 12-dimensional F-theory. Leaving open the question, whether such a theory (with 12-dimensional dynamics) really exists, this theory has a sound foundation as a particular class of type IIB compactifications with varying dilaton. Such compactifications are classically impossible and can only be defined by using an $SL(2,\mathbb{Z})$ self-duality of the type IIB string, which is geometrized by introducing two extra spacetime dimensions. Compactifications of F-theory are defined by elliptically fibered Calabi-Yau varieties.

A very interesting duality was discovered between F-theory compactified on an elliptic K3 manifold and the heterotic string compactified on the two-torus, i.e. the generic fiber. This basic duality can be lifted to higher dimension of the compactification spaces by application of fiberwise[5] dualities. This duality is

[3] The least obvious step in their construction is the GSO projection [GSO76, GSO77], the precise form of which distinguishes between the two types of theories.

[4] The letter M later obtained an additional interpretation, when matrix theories were suggested as a description for the unifying theory.

[5] Extrapolating dualities fiberwise is a somewhat dangerous process. At least, the interplay between local and global properties of holomorphic objects forces one to take into account global properties of fibered spaces, e.g. monodromy. But even if some assumptions (like the adiabatic argument "slowly varying") are unjustified, such arguments can provide valuable ideas.

particularly appealing, since it offers the perspective of trading vector bundles for “just” an additional complex dimension of the compactification space.

When studying F-theory compactifications, the special role of the elliptic fiber (“frozen torus”) may be troublesome. This special treatment may be circumvented by using another duality, which connects F-theory on a Calabi-Yau variety Y further compactified on $\mathbb{S}^1$ with M-theory on Y. M-theory in turn is believed to be the strong coupling limit of the type IIA string.

In order to obtain space-time dimension four, the heterotic string is compactified on a Calabi-Yau threefold and F-theory on an elliptic Calabi-Yau fourfold. For a fiberwise extension of the basic duality to a heterotic string theory, the Calabi-Yau fourfold should be elliptic K3 fibered and the compactification space of the dual heterotic string theory should be obtained by replacing the generic K3 fiber with the generic elliptic fiber.

In fact, elliptic K3 fibrations are more than just a tool for conjecturing dualities. In [AL96] it was shown, that the compactification space of a type IIA string theory must be a K3 fibration if it is to be dual to a heterotic string theory[6]. Note, that when talking about K3 (elliptic, ...) fibrations, I always mean a fibration with *generic* fiber a K3 surface (elliptic curve, ...).

It is very desirable to have as many explicitly calculable examples for the objects involved in such dualities as possible. Such examples are certainly valuable for discovering limitations of duality hypotheses and might also lead to new unexpected connections. In addition, the knowledge of sufficiently many explicit examples might some day help in proving dualities by application of deformation arguments.

A very powerful tool for constructing interesting yet manageable examples is given by toric geometry. When toric geometry first was made widely known to physicist by the work of V. Batyrev [Bat94], it provided the first example of a large class of threefold varieties which was closed under mirror symmetry, namely the class of Calabi-Yau hypersurfaces in toric varieties corresponding to four-dimensional reflexive polyhedra. This class is a superclass of three-dimensional quasismooth hypersurfaces in weighted projective spaces. The mirror duality is beautifully expressed as the combinatorical duality between reflexive polyhedra. When other dualities than mirror symmetry were discovered, there was a hope to find equally beautiful combinatorical representations.

Unfortunately, equally self-contained pictures do not emerge in more general applications of toric geometry. Toric descriptions for the vector bundles involved in heterotic string compactifications did not prove to be very useful. The toric mirror construction via dual Gorenstein cones [BB97] describing complete intersection configurations is a beautiful generalization of the mirror duality for hypersurfaces, but the class of varieties corresponding to reflexive Gorenstein cones is much less general than complete intersection subvarieties of products of weighted projective spaces[7].

[6] One has to be a little bit clearer about what is meant by “heterotic string theory”. In this context, one demands the theory to have a purely geometric phase.

[7] In particular, the complete intersection Calabi-Yau varieties in pairs of 0-2 mirror dual heterotic theories as calculated in [BSW97] almost never are *both* described by reflexive Goren-

Turning to four-dimensional spaces used for F- and M-theory compactifications, the class of transverse hypersurfaces in weighted projective spaces unfortunately cannot be embedded in the class of hypersurfaces in toric varieties corresponding to reflexive polyhedra. But giving up hopes of generality, the class of four-dimensional Calabi-Yau hypersurfaces in toric varieties corresponding to reflexive polyhedra provides a *very*[8] large class of examples. These examples allow the same degree of explicit control as their lower dimensional counterparts, in particular as far as singularity structures and counting of moduli are concerned.

As noted above, fibration structures with lower dimensional Calabi-Yau spaces as generic fibers are very important properties of compactification spaces. This is even more true in four than in three dimensions. The question arose, whether such fibrations can be studied in a framework as simple as the description of hypersurfaces in toric varieties. Fortunately, the answer is yes and was found in [AKMS97]. Since the original treatment is sufficiently imprecise and original hopes about the generality of the construction proved to be wrong, I will review the construction in section 3.

In the context of the aforementioned $Het \leftrightarrow F$ duality, elliptic K3 fibered Calabi-Yau hypersurfaces in five-dimensional toric varieties are of particular interest. Due to the lack of classification of the corresponding polyhedra, a class of such varieties was constructed in [KLRY98] and studied in e.g. [BCOG00]. Although tailormade for precisely this application, even in the study of these varieties problems were encountered concerning the unfulfilled hopes conjectured in [AKMS97] — some of the conjectured fibrations simply did not exist. This problem was circumvented by passing to other polyhedra, where the problems did not arise[9].

In this paper, a large class of examples will be exhibited by searching for fibration structures rather than constructing them. Instead of passing to more agreeable polyhedra[10], the problem concerning the existence of fibrations will be treated by allowing for more general fibrations. In particular, exceptional fibers with higher than generic dimension are allowed.

stein cones.

[8] The set of inequivalent five-dimensional reflexive polyhedra, though not completely known, is certainly larger than the set of five-dimensional weighted projective spaces allowing for transverse hypersurfaces. Although the class of families of varieties corresponding to the former is not a superset of the latter, this is already obvious from trivial extensions of the 473,800,776 [KS00] four-dimensional reflexive polyhedra compared to 1,100,055 weight sets [LSW99].

[9] There are examples where inequivalent polyhedra give rise to equivalent families of hypersurfaces. It was observed, that the Hodge numbers do not change in the process of passing to the alternative polyhedron. This suggests, but does not prove, that the corresponding families of hypersurfaces are identical.

[10] The existence of suitable replacements is not always clear.

2 Notational conventions

I assume that the reader is familiar with basic properties of toric varieties, reflexive polyhedra and Calabi-Yau hypersurfaces in toric varieties. Since there are different notational conventions in use, I will shortly summarize the conventions I use throughout this paper.

In order not to unnecessarily clobber notation, I will from time to time refrain from notationally distinguishing between different, but closely related, objects.

Almost all varieties encountered in this paper will be complex varieties. Therefore, I will refrain from notationally emphasizing this and e.g. simply write $\mathbb{P}^n$ instead of $\mathbb{CP}^n$ for the complex n-dimensional projective space.

For toric varieties, I start (as usual) with dual lattices $N \cong \mathbb{Z}^n, M = N^\star$. Even if not explicitly stated, I will always choose fixed isomorphisms $N \cong \mathbb{Z}^n \subset \mathbb{R}^n \cong N_\mathbb{R}$ in order to identify linear maps (lattice morphisms) and real (integer) matrices[11]. This enables me to simply write N for both the lattice and its real extension.

Fans live in N and will be denoted by the letter Σ (no lattice index due to the fixed isomorphisms). $|\Sigma|$ denotes the support of Σ and $\Sigma^{(d)}$ the subset of Σ consisting of cones of dimension d. The toric variety corresponding to a fan Σ will be written as X_Σ and $\mathbb{T} \subset X_\Sigma$ denotes the open algebraic torus contained in X_Σ.

Pairs of dual Polyhedra will be denoted (∇, Δ), $\nabla \subset N$, $\Delta \subset M$. This convention will also be used for nonreflexive (e.g. non-integral) polyhedra. Inner face normals will always be normalized to have scalar product -1 with the corresponding hypersurface and can thus (for nonreflexive polyhedra) fail to be integral.

Given a pair of reflexive Polyhedra (∇, Δ), X_Δ will (depending on the context) denote either the set of toric varieties given by fans over triangulations of $\partial\nabla$ or a particular element of it. Usually (but not always) the triangulations will be maximal.

The Calabi-Yau hypersurface defined by the set Z_p of zeroes of a generic section p of the anticanonical bundle on X_Δ will be called a **toric Calabi-Yau hypersurface** (although it is not a toric variety itself). As usual, the divisors of Z_p given by pullbacks of toric divisors of X_Δ are called **toric divisors**. Divisors not obtained in this way are called **nontoric**[12].

It is well known, that divisors of X_Δ corresponding to rays over points in the interior of a hypersurface of ∇ do not intersect the generic Calabi-Yau hypersurface. There are different ways to deal with such points, in particular when studying different triangulations of $\partial\nabla$ corresponding to different cones in the secondary fan. One way is to omit them introducing singularities in the toric varieties, which do not not meet the Calabi-Yau hypersurfaces. When studying

[11] This is not a standard convention, because for studying abelian quotients of toric varieties it is more favourable to use different lattices inside the same real extension.

[12] Usually, nontoric divisors are equivalent to sums of irreducible components of toric divisors. This statement may fail in the case of K3 hypersurfaces, c.f. section 4.

Kähler moduli spaces and secondary fans, this is the favoured approach, since one directly obtains the correct dimension for the space of Kähler forms on the hypersurface[13]. Another approach is to include the points in order to obtain maximal triangulations[14] of $\partial\nabla$. Then, the additional degrees of freedom in the choice of Kähler class lie in the kernel of the pullback to the hypersurface. Yet another way is to completely omit the maximal dimensional cones from the fan Σ, which sacrifices compactness of the toric varieties[15]. I will write $\partial^1\nabla$ for the union of faces of ∇ with codimension ≥ 2.

I will freely switch between the classic approach to toric varieties using fans and lattices and the holomorphic quotient approach [Aud91, Bat93, Mus94, Cox95]. The section p defining the hypersurface will be written either in terms of torus characters or homogenous monomials. For a description of how to translate between the two representations, refer to e.g. [Cox97].

3 Calabi-Yau fibered toric hypersurfaces

The main subject of this paper will be (sub-) families of Calabi-Yau hypersurfaces in toric varieties, the generic members of which carry a fibration structure with generic fiber again a Calabi-Yau hypersurface in some lower dimensional toric variety.

As in the case of the hypersurfaces themselves, one is interested in a framework, which allows to deduce properties of the generic fibration structure from the polyhedron describing the embedding toric variety alone. Such a framework was discovered by the authors of [AKMS97]. Put into a nutshell, their statement was as follows: Such a fibration structure exists, whenever the dual reflexive polyhedron ∇ of the embedding toric variety contains a reflexive subpolyhedron $\nabla^{(f)} \subset \nabla$. The generic fiber then is a Calabi-Yau hypersurface in $X_{\Delta^{(f)}}$.

As there is no obvious way of making the above statement both precise and true[16], I will briefly elaborate on this subject.

Let $\nabla = \Delta^\star$ be a reflexive polyhedron, X_Δ the toric variety corresponding to some projective[17] triangulation of $\partial^1\nabla$ and $Z_f \subset X_\Delta$ the Calabi-Yau hypersurface given by the set of zeroes of the (generic) section p of the anticanonical bundle on X_Δ. The structures we are looking for are fibrations

$$\begin{array}{ccc} X_{\Delta^{(f)}} & & \\ \downarrow & & \\ X_\Delta & \xrightarrow{\pi} & B, \end{array}$$

[13] Nevertheless, different cones in the secondary fan do not necessarily correspond to topologically different phases of the hypersurface. This happens, whenever the corresponding triangulations do not differ on faces of codimension ≥ 2.

[14] This is technically appealing when e.g. studying K3 hypersurfaces.

[15] I will use this approach for the toric prefibrations defined later in this paper.

[16] In addition, both the original paper as well as follow-ups remain unclear about some details of the fibration structures.

[17] Projective triangulations allow strictly convex functions, which are affine on each simplex. Precisely under this condition does the toric variety allow a Kähler form.

where the generic fiber $X_{\Delta^{(f)}}$ is given by a pair $(\nabla^{(f)}, \Delta^{(f)})$ of reflexive polyhedra and the base B of the fibration is again a toric variety. Of course, I demand π to be a morphism of toric varieties. In addition, I require the intersections $Z_p \cap X_{\Delta^{(f)}}$ to be Calabi-Yau hypersurfaces over generic points in the base B. The toric fibration alone (i.e. without restricting to Z_p) will be called a **toric Calabi-Yau prefibration** and will be denoted $(X_{\Delta^{(f)}}, X_\Delta, B)$.

Let the base B be given by a fan Σ_b. Corresponding fans, polyhedra and lattices will be marked with the same letters, e.g. Σ for the fan over some projective triangulation of $\partial^1\nabla$, $\Sigma^{(f)}$ for the subfan in $N^{(f)}$, $\Delta^{(f)} \subset M^{(f)}$, and so on[18].

As a toric morphism, π is given by a lattice morphism $\hat{\pi} : N \twoheadrightarrow N_b$ with the usual conditions enforced by continuity: $\forall \sigma \in \Sigma\ \exists \sigma_b \in \Sigma_b :\ \hat{\pi}(\sigma) \subset \sigma_b$.

It should be noted at this point, that I have fixed a fan Σ corresponding to a subfamily of X_Δ. In general, the corresponding triangulation[19] cannot be replaced by some other triangulation without violating the conditions on the lattice morphism $\hat{\pi}$ to define a morphism of toric varieties. For given ∇, I will later study whether appropriate triangulations exists at all.

Note, that there is no need to demand the morphism π to be defined on torus orbits of X_Δ, which do not meet the hypersurface Z_p. I avoid any potential problems in this respect by using fans Σ without maximal dimensional cones, i.e. fans over triangulations of $\partial^1\nabla$.

The lattice morphism $\hat{\pi}$ must be surjective, because otherwise it could be factored into a surjective morphism followed by an embedding. The first part would then define a fibration in its own right (with base given by the same fan but courser lattice), while the second describes an abelian quotient map from the intermediate base to the original one. The generic fiber of the original fibration would then be the disjoint union of the fibers over multiple points in the intermediate base and hence could not be of the required type.

Since B is supposed to be the base of the fibration, π must be surjective, or in terms of the fans $\hat{\pi}(|\Sigma|) = |\Sigma_b|$. This includes the commonly used case $\Sigma_b = \hat{\pi}(\Sigma)$, but does not imply it. As a simple consequence, Σ_b must be complete. This would be obvious if I considered complete fans Σ. Here, it follows from

Lemma 3.1 *Let $\sigma \subset \mathbb{Q}^N$ be a strongly convex N-dimensional rational cone, $\sigma' \subset \mathbb{Q}^n$ a convex rational n-dimensional cone and $\pi : \mathbb{Q}^N \to \mathbb{Q}^n$ a linear map with $\pi(\sigma) = \sigma'$. Let $\Sigma^{(k)} := \{\tau \,|\, \tau \text{ is a face of } \sigma\ \wedge\ \dim\tau = k\}$. Then $\sigma' = \bigcup_{\tau\in\Sigma^{(n)}} \pi(\tau)$.*

Proof: I will prove $N > n \Rightarrow \sigma' = \bigcup_{\tau\in\Sigma^{(N-1)}} \pi(\tau)$, which is equivalent to the assertion by complete induction. I must show, that any $\pi(x)$, $x \in \sigma$, has a preimage on the boundary of σ. Since $N > n$, $\dim\ker\pi \geq 1$ and $\exists\, 0 \neq y \in \ker\pi$. Hence, $\forall\, t \in \mathbb{Q} : \pi(x + ty) = \pi(x)$. Since σ is strongly convex, the line $x + ty$ must intersect the boundary of σ. □

[18] Also remember our convention of writing $N^{(f)}$ for both the lattice and its real extension.

[19] More generally, Σ will not always stem from a triangulation, but rather from a more general polyhedral partition. In such a case, any simplicial refinement of Σ is also compatible with Σ_b.

One easily calculates[20] the preimage of $\mathbb{T}_b$ to be the open subvariety of X_Σ given by the fan $\Sigma^{gen} = \{\sigma \cap \ker \hat{\pi} \mid \sigma \in \Sigma\}$. Now $\Sigma^{gen} \subset \Sigma$, because strong convexity of the cones in Σ_b implies that $\sigma \cap \ker \hat{\pi}$ must be a face of σ for any cone $\sigma \in \Sigma$.

I set $N^{(f)} := N \cap \ker \hat{\pi}$ and $\Sigma^{(f)} = \{\sigma \cap \ker \hat{\pi} \mid \sigma \in \Sigma\}$. N can be split[21] as $N = N^{(f)} \oplus \tilde{N}_b$. Obviously, $\tilde{N}_b \cong N_b$. We have $M = \left(N^{(f)}\right)^\star \oplus \left(\tilde{N}_b\right)^\star$ with $\left(\tilde{N}_b\right)^\star \cong M_b$ and write $M^{(f)} := \left(N^{(f)}\right)^\star$, $\pi_M : M \to M^{(f)}$ for the projection to the second factor and $L_M : M^{(f)} \to M$ for the lift induced by our choice of $\tilde{N}_b$. The well known duality between intersections and projections here manifests itself as $\Delta^{(f)} = \pi_M(\Delta)$.

With this notation,

$$\pi^{-1}(\mathbb{T}_b) = X_{\Sigma^{(f)}} \times \mathbb{T}_b$$

and π operates by projecting to the second factor. The fan for the generic fiber of the toric prefibration is thus given by $\Sigma^{(f)}$. If the generic fiber is to be a member of the family $X_{\Delta^{(f)}}$, we must have $\nabla^{(f)} = \nabla \cap \ker \hat{\pi}$. Such an embedding $\nabla^{(f)} \hookrightarrow \nabla$ will be called a **virtual** toric Calabi-Yau prefibration.

Remark 3.2 *It is important, that we consider* generic *fibrations, i.e. we allow the fibers to degenerate over subvarieties of strictly smaller dimension. This is true not only for the Calabi-Yau fibrations, but also for the prefibrations. The statement found in [KS98a], that the toric prefibrations are subject of an exercise in [Ful93, p.41] is wrong, as in this reference only locally trivial fibrations are considered and strong conditions on the fans are derived. These conditions are not met by (most of) our prefibrations. In fact, interesting physics (like nonabelian gauge groups) emerges precisely in those cases, where the fibrations degenerate. As a simple example for the difference, consider $\mathbb{P}^1$ fibrations over $\mathbb{P}^1$: The only locally trivial fibrations are the Hirzebruch surfaces, while any blowup thereof yields a generic fibration with bouquets of $\mathbb{P}^1$s as fibers over the poles.*

3.1 Calabi-Yau fibers

Let us assume we have found a toric Calabi-Yau prefibration $(X_{\Delta^{(f)}}, X_\Delta, B)$ as discussed above. Let Z_p be the Calabi-Yau hypersurface in X_Δ given by the set of zeroes of a generic global section p of the anticanonical bundle. We will see, that the prefibration makes Z_p a fibration over the same base. The generic fibers of this fibration are indeed Calabi-Yau varieties.

We first take a closer look at the intersections of Z_p with the generic fibers of the prefibration. To this end, we write the generic section p as

$$p = \sum_{m \in \Delta \cap M} a_m \chi^m,$$

[20] using e.g. $\mathbb{C}$-valued points

[21] c.f. appendix A for a proof. Note that the splitting is not unique: We can choose an arbitrary lift $\tilde{N}_b$ of N_b.

where $a_m \in \mathbb{C}$ are generic coefficients and χ^m is the global section represented by the torus character corresponding to $m \in M$. This can be split as follows:

$$
\begin{aligned}
p &= \sum_{m \in \Delta \cap M} a_m \chi^m \\
&= \sum_{m_f \in \Delta_f \cap M^{(f)}} \sum_{m \in \Delta \cap M:\, \pi_M(m)=m_f} a_m \chi^m \\
&= \sum_{m_f \in \Delta_f \cap M^{(f)}} \sum_{m' \in \left(\tilde{N}_b\right)^\star : L_M(m_f)+m' \in \Delta} a_m \chi^m \\
&= \sum_{m_f \in \Delta_f \cap M^{(f)}} \left(\sum_{m' \in \left(\tilde{N}_b\right)^\star : L_M(m_f)+m' \in \Delta} a_m \chi^{m'} \right) \chi^{L_M(m_f)}. \qquad (1)
\end{aligned}
$$

Let $\{n_i \in N^{(f)}\}$ denote the primitive generators of the rays in $\left(\Sigma^{(f)}\right)^{(1)}$. Introducing homogenous coordinates $X_i, i \in I$ on the fiber space and replacing torus characters by the corresponding homogenous monomials, the restriction of p to $X_{\Sigma^{(f)}} \times \mathbb{T}_b$ can be rewritten as

$$
p \sim \sum_{m_f \in \Delta_f \cap M^{(f)}} \left(\sum_{m' \in (N_b)^\star : L_M(m_f)+m' \in \Delta} a_m \chi^{m'} \right) \prod_{i \in I} X_i^{\langle n_i, m_f \rangle + 1}. \qquad (2)
$$

Over any point in the base $\mathbb{T}_b$ (1) or (2) is a global section of the anticanonical bundle on the fiber space, whose coefficients are given by Laurent polynomials

$$
A_{m_f} := \sum_{m' \in \left(\tilde{N}_b\right)^\star : L_M(m_f)+m' \in \Delta} a_m \chi^{m'}
$$

in the base torus coordinates. At least over generic points in $\mathbb{T}_b$ one has $p \not\equiv 0$ and thus the intersection of the generic fiber with the Calabi-Yau hypersurface is indeed a Calabi-Yau variety.

We still need to check, whether X_{Σ_b} is the base not only of our toric prefibration, but also of the Calabi-Yau fibration, i.e. whether π remains surjective when restricted to the Calabi-Yau hypersurface Z_p. The calculations done so far show, that $\pi(Z_p)$ is a Zariski open subset of X_b. As Z_p is compact and π is continuous, $\pi(Z_p)$ is compact as well[22]. Hence, $\pi(Z_p) = B$.

3.1.1 How generic are the fibers?

Most of the theorems yielding properties of Calabi-Yau hypersurfaces from those of the embedding toric variety demand the defining section to be generic. Hence, it must be ensured that the (generically chosen) section p remains sufficiently generic when restricted to the prefibration fibers.

[22] Here we use the analytic topology.

Obviously, the only case for which a coefficient A_{m_f} will not yield generic values over generic points in the base is $A_{m_f} \equiv 0$. Unfortunately, this can happen. For a simple example consider the three-dimensional reflexive Polyhedron Δ_3 with vertices $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1\\-2\\-1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$ and inner face normals $\left(\begin{smallmatrix}-1\\-1\\4\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1\\-1\\0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1\\3\\-4\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}3\\-1\\0\end{smallmatrix}\right)$. Projecting along the direction of the third coordinate, we obtain the reflexive triangle with vertices $\binom{1}{0}$, $\binom{0}{1}$ and $\binom{-1}{-2}$. Apart from the vertices and the origin, this contains one additional integer point, namely $\binom{0}{-1}$, which has no integer preimage in Δ_3.

The good news is, that this does not endanger $\Sigma^{(f)}$-regularity of the generic Calabi-Yau fiber, since the Bertini-type argument already works when only using the monomials for the vertices and the origin contained in $\Delta^{(f)}$. These in turn always have integer preimages due to simple convexity arguments.

Unfortunately, for deducing the Picard lattice of a generic K3 fiber, Σ-regularity is not sufficient [Roh04]. If the projection map

$$\pi_M : \Delta \cap M \longrightarrow \Delta^{(f)} \cap M_f$$

is not surjective, toric methods might only reveal a sublattice of the true generic Picard lattice[23].

3.2 Obstructions

As remarked above, the existence of a reflexive subpolyhedron in general is not sufficient for finding toric Calabi-Yau prefibrations (and corresponding toric Calabi-Yau fibrations). The additional requirement is the existence of fans Σ, Σ_b, for which the lattice morphism dictated by the subpolyhedron defines a toric morphism.

Such fans always exist, if the codimension of $\nabla^{(f)} \subset \nabla$ is one. In this case, $\nabla^{(f)}$ separates $\partial\nabla$ into two halves, which can always be triangulated separately. Any such triangulation defines a fan compatible with the fan of $\mathbb{P}^1$, which is the only possibility in this case.

Let us now turn to the case of higher codimension. If we consider some given triangulation of ∇ resp. the fan Σ given by it, it is easy to check whether there exists a compatible fan Σ_b. We start with the set of cones given by $\tilde{\Sigma}_b := \hat{\pi}(\Sigma)$. In most cases, this will not be a fan, but since we want $\hat{\pi}$ to define a toric morphism, the fan Σ_b must satisfy

$$\forall \tilde{\sigma} \in \tilde{\Sigma}_b \; \exists \sigma \in \Sigma_b : \; \tilde{\sigma} \subset \sigma.$$

If $\tilde{\Sigma}_b$ contains a cone, which is not *strongly* convex, a compatible fan obviously cannot exist. Otherwise, $\tilde{\Sigma}_b$ can only fail to be a fan in its own right due to the existence of cones, whose intersection is not a common face. We can thus

[23] This loss of knowledge is certainly not satisfactory, but it is not easily resolved. Lower bounds on the enhanced Picard lattices could be obtained e.g. by studying enlarged automorphism groups of the hypersurfaces.

construct the finest possible Σ_b by enlarging cones in $\tilde{\Sigma}_b$ (and throwing away subcones).

Since there are only finitely many triangulations of $\partial^1\nabla$, a simple approach to check the existence of compatible fans is given by simply trying to construct a compatible Σ_b for each of them. Unfortunately, this approach is computationally feasible only for the very simplest cases (for $\dim\nabla > 3$ the number of possible triangulations is always finite, but almost always huge[24]).

If one puts some weak restrictions on the fibrations though, it is possible to directly construct *some* compatible triangulation, if any exists. The restrictions used and the methods for constructing the triangulations will be discussed in section 5.3.

3.3 Multifibrations

In this section A-fibered B-fibrations X will be discussed. The obvious interpretation of the above term is that X is a fibration, whose generic fiber is an object of type B, which itself is a fibration with generic fiber A. One could also define it as a space X, which is both an A- and a B-fibration. I will use the term in a more restrictive way and demand both. Phrased differently, I demand that the B- fibration X also is an A-fibration and the fibration structures are compatible. In order to make the last statement precise, I will introduce a third fibration as follows[25].

Definition 3.3 *An A-fibered B-fibration is a chain of morphisms*

$$X \xrightarrow{\pi_1} Y \xrightarrow{\pi_2} Z$$

with the following properties:

(i) $X \xrightarrow{\pi_1} Y$ *is an A-fibration. A is called the small generic fiber and Y the large base.*

(ii) $Y \xrightarrow{\pi_2} Z$ *is a fibration with generic fiber F. Z is called the small base and F the intermediate base.*

(iii) $X \xrightarrow{\pi_2\circ\pi_1} Z$ *is a B-fibration. B is called the generic large fiber.*

Note, that these conditions force the generic fibers of $X \xrightarrow{\pi_2\circ\pi_1} Z$ to be A-fibrations over the intermediate base F.

In order to simplify the discussion, I will now specialize to the case of elliptic K3 fibered Calabi-Yau hypersurfaces (either threefolds or fourfolds) in toric

[24] If one wants to enumerate all triangulations, the computational complexity is largely reduced by only considering regular triangulations, i.e. triangulations for which the resulting varieties are Kähler. One can then calculate the secondary polytope instead of a brute force enumeration. Although this reduces the complexity of the enumeration from NP-hard to polynomial, the number of regular triangulations is still huge in most cases.

[25] The structures are all very simple, but can be confusing on first sight. The latter is reflected in the fact, that no good comprehensive diagrammatic notation for multifibrations seems to exist.

varieties. With the above notations, the small fiber $A = T^2$ is an elliptic curve, the large fiber B is K3, and the immediate base F is $\mathbb{P}^1$.

In complete analogy to the case of a single fibration, a **toric elliptic K3 fibration** is the restriction to a generic Calabi-Yau hypersurface of a **toric elliptic K3 prefibration**. The latter consists of a virtual prefibration

$$\nabla_{ell} \subset \nabla_{K3} \subset \nabla$$

together with compatible base fans $\Sigma_{b,ell}, \Sigma_{b,K3}$ and a compatible triangulation of $\partial^1\nabla$. For the prefibration, A is $X_{\Delta_{ell}}$, B is $X_{\Delta_{K3}}$ and F is again $\mathbb{P}^1$.

To give the notation intuitive meaning, we rename π_1 to π_e and define $\pi_K := \pi_2 \circ \pi_e$.

As in the case of a single fibration one can split[26] $N = N_e \oplus \mathbb{Z} \oplus N_{b,K}$ with $N_K = N_e \oplus \mathbb{Z}$ and $N_{b,e} = \mathbb{Z} \oplus N_{b,K}$ and obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & N_e & \hookrightarrow & N_K & \overset{\hat{\pi}_{e,K}}{\twoheadrightarrow} & \mathbb{Z} & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & N_e & \hookrightarrow & N & \overset{\hat{\pi}_e}{\twoheadrightarrow} & N_{b,e} & \to & 0 \\
 & & & & \hat{\pi}_K \downarrow & & \downarrow \hat{\pi}_2 & & \\
 & & & & N_{b,K} & = & N_{b,K} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (3)$$

with exact rows and columns.

3.4 Algorithms for finding virtual fibrations

Given a collection of D-dimensional reflexive polyhedra, one would like to find all toric Calabi-Yau fibration structures carried by generic members of the corresponding families of toric Calabi-Yau hypersurfaces. As a first step towards this aim, one obviously has to find all virtual fibration structures, i.e. reflexive subpolyhedra.

3.4.1 Enumerating sublattices

The most simple way to find reflexive subpolyhedra of a given D-dimensional polyhedron $\nabla \subset N_{\mathbb{R}}$ uses the fact, that any d-dimensional lattice subpolyhedron $\nabla^{(f)}$ defines the d-dimensional lattice subspace $N_{\mathbb{R}}^{(f)}$ spanned by its vertices, which in turn are integer points of ∇. Thus, the potential lattice subspaces can be enumerated via d-tuples of integer points of ∇. These subspaces can then be intersected with ∇ yielding a (not necessarily integer) polyhedron $\nabla^{(f)}$, which then can be checked for reflexivity. The check for reflexivity can be performed as follows:

[26] As before, the splitting is not uniquely determined.

Consider the lattice subspace $N_{\mathbb{R}}^{(f)}$ spanned by integer points $n_1, \ldots, n_d$ of ∇. We first calculate[27] a base

$$\{k_1, \ldots, k_{D-d}, b_1, \ldots, b_d\} \tag{4}$$

of M, where

$$\{k_1, \ldots, k_{D-d}\} \tag{5}$$

is a base of $N^{(f)} := (N_{\mathbb{R}}^{(f)})^{\perp} \cap M$. Let

$$K := (k_1 \cdots k_{D-d})\,,\ c_N := (b_1 \cdots b_d)^t\,,\ B := \begin{pmatrix} K^t \\ c_N \end{pmatrix}$$

and π_M be the last d rows of $\left(B^{-1}\right)^t$. Note, that the columns of π_M^t form a base of the sublattice $N^{(f)}$. Using this base and its dual base for $M^{(f)}$, π_M is the matrix of the projection $\pi_M : M \to M^{(f)}$. We have $n_f \in N^{(f)} \Leftrightarrow K^t n_f = 0$ and the n_f are transformed to the base π_M^t of the sublattice by the matrix c_N.

Using K^t we can immediately determine all elements of $N^{(f)}$ among the integer points of ∇. This information can be used to avoid multiply checking the same subspace defined by different tuples of points. In addition, it provides a first simple check for reflexivity. If $\nabla^{(f)}$ is to be reflexive at least its vertices and the origin are integer points. Hence, we can immediately dismiss the subspace if we do not find at least $d+2$ points.

Now let $\{m_i\}$ be the set of inner face normals of ∇, i.e. the vertices of $\Delta = \nabla^{\star}$. Obviously, $\{\pi_M m_i\}$ generates $\Delta^{(f)} := (\nabla^{(f)})^{\star}$ as a convex set. From the inner point criterion for reflexivity, none of the $\pi_M m_i$ can have integer length > 1 if $\nabla^{(f)}$ is to be reflexive.

Assume, that all of the above tests are passed. The final test for reflexivity can then be performed in either of two ways:

- One calculates the convex hull of $\{\pi_M m_i\}$ in terms of its hypersurfaces. If (and only if) all inner face normals are integer[28], $\Delta^{(f)}$ is reflexive and so is $\nabla^{(f)}$.

- Denote the convex hull of the integer points of $\nabla^{(f)}$ by C. $\nabla^{(f)}$ is reflexive if and only if $C = \nabla^{(f)} = (\Delta^{(f)})^{\star}$. One calculates the inner face normals of C, which must be integer for $\nabla^{(f)}$ to be reflexive. In this case, though, this is not sufficient. C could be smaller than $\nabla^{(f)}$, in which case $C^{\star}$ is larger than $\Delta^{(f)}$. If $\nabla^{(f)}$ is reflexive, the vertices of $C^{\star}$ are the vertices of $\Delta^{(f)}$ and thus are elements of $\{\pi_M m_i\}$. If it is not, there must be some vertex of $C^{\star}$ not contained in $\Delta^{(f)}$ and hence not an element of $\{\pi_M m_i\}$.

[27] One can use Gaussian elimination to calculate the kernel of $(n_1 \cdots n_d)$. This is a lattice subspace of M, because $(n_1 \cdots n_d)$ is an integer matrix. Then one can use the methods presented in appendix A to calculate the bases (5) and (4). If the kernel is not $(D-d)$-dimensional, the points are linearly dependent and the result might thus be reflexive, but obviously not d-dimensional.

[28] Remember, that face normals are normalized to have inner product -1 with the face.

The most expensive part in either of the two ways is calculating the convex hull (the inner face normals) of a finite point set. One should thus choose between the two ways depending on the size of the corresponding point set.

Anyway, the check for reflexivity does not pose a serious computational threat in any explicit example. The problem with the algorithm is rather given by the number of subspaces to check, which badly scales with the number n of integer points in ∇ approximately like $\binom{n}{d} \sim n^d$. For fourfold polyhedra this poses a serious threat, since one easily finds examples of polyhedra with 10^4 - 10^5 integer points.

The problem is obviously less serious, if one does not need to enumerate all lattice subspaces, e.g. by fixing some subspace. I actually used sublattice enumerations for finding reflexive three-dimensinal subpolyhedra, which are superpolyhedra of a given two-dimensional subpolyhedron. The number of subspaces one has to check in such a setting for this task only scales linearly with the number of integer points.

3.4.2 Making use of classifications

Instead of directly searching for reflexive subpolyhedra, one can also use the dual picture and look for projections $\pi_M : \Delta \twoheadrightarrow \Delta^{(f)}$. This was used by the authors of [AKMS97], who looked for reflexive faces of Δ and projections to these faces. I did not follow this path, because it further reduces the generality of the fibrations one finds. More seriously, it inherently carries the risk of missing some of the most interesting structures.

One cannot simply enumerate all possible projections except by enumeration of subspaces as in the preceding section (which would mean no gain), because the kernel of the projection map does not have to generated by integer points of Δ.

For cases, where a classification of the reflexive polyhedra in d dimensions is at hand, the projection approach nevertheless provides an algorithm, which is almost[29] independent of n. The main idea is to search for projections $\pi_M : \Delta \twoheadrightarrow \Delta^{(f)}$ for all possible reflexive images $\Delta^{(f)}$.

Let $(\Delta = \nabla^\star, M)$ be as before and $(\Delta^{(f)}, M_f)$ a reflexive polyhedron of dimension d. We need to know, whether there is a lattice projection $\pi_M : M \twoheadrightarrow M_f$, which maps Δ to $\Delta^{(f)}$.

If there is such a map, it will map the integer points of Δ onto the integer points of $\Delta^{(f)}$. In particular, it will map the vertices of Δ to integer points of $\Delta^{(f)}$ and due to linearity all vertices of $\Delta^{(f)}$ will be images of vertices of Δ. We thus choose D linearly independent vertices $m_1, \ldots, m_D$ of Δ. Their images completely determine π_M by linearity. Hence, we can walk through all maps $\{m_1, \ldots, m_D\} \to \{\text{integer points of } \Delta^{(f)}\}$, reconstruct π_M for each map and check whether it fulfills the conditions. In detail, we check if

- π_M is integer.

[29] Almost, because it does depend on the number of hypersurfaces of ∇, which is not statistically independent.

- π_M has maximal rank.
- π_M is surjective: There must be a primitive sublattice of M isomorphic to M_f. The inverse image of our base for M_f under the isomorphism can be completed to a base for M. Using this base, the matrix of π_M is obtained by omitting rows from the unit matrix. Hence, the rows of π_M must be completeable to a basis for M. The latter can be checked using algorithm A.2.
- π_M maps the vertices of Δ into $\Delta^{(f)}$.
- one can find preimages of all vertices of $\Delta^{(f)}$ among the vertices of Δ.

Of course, the number of mappings one has to check could be significantly reduced by using known automorphisms of Δ. Since one usually does not know these beforehand and calculating them is computationally expensive, I did not implement this.

Properties of members of the classification list, on the other hand, are well known beforehand and can be used to optimize the algorithm. In particular, one can

a) use automorphisms of the $\Delta^{(f)}$ in order to reduce the number of potential mappings and

b) use embeddings $\Delta_1^{(f)} \hookrightarrow \Delta_2^{(f)}$ to search for projections to $\Delta_1^{(f)}$ while searching those to $\Delta_2^{(f)}$.

Note though, that (a) and (b) partially exclude each other.

4 Elliptically fibered toric K3 surfaces

The basic duality between heterotic string and F-theory is between F-theory compactified on an elliptically fibered K3 and the heterotic string compactified on the generic fiber. This makes elliptically fibered K3 surfaces an interesting object of study.

As mentioned in the introduction, the fiberwise extension of the basic duality leads to a duality between F-theory on an elliptic K3 fibered Calabi-Yau variety and the heterotic string on an elliptic Calabi-Yau threefold.

The perturbative gauge group of the heterotic string theory dual to F-theory on an elliptic K3 fibered Calabi-Yau variety is then encoded in the generic K3 fiber and global monodromy arguments. Thus, a good knowledge of elliptically fibered K3 surfaces will also help in reading off properties of fibrations.

Elliptically fibered K3 surfaces are the easiest examples of toric Calabi-Yau fibrations. In addition to the low dimensionality, the codimension of the fiber is one. Hence, any virtual Calabi-Yau prefibration gives rise to an elliptic fibration.

In the following sections I will often use properties of the Picard lattice of a generic toric K3 surface, including its intersection form. A full derivation of the formulae used to calculate this lattice can be found in [Roh04]. For the reader's convenience, they are summarized in appendix B.

4.1 The class of the fiber

The definition of F-theory on a Calabi-Yau manifold requires it to be elliptically fibered, and for taking the F-theory limit one needs a section of the fibration

$$\begin{array}{ccc} E & & \\ \downarrow & & \\ Y & \longrightarrow & B. \end{array}$$

The image of the section $s : B \to Y$ is an effective divisor having intersection number 1 with the generic fiber. Specializing to toric fibrations I will only consider divisors of Y, which are sums of pullbacks of toric divisors or irreducible components thereof. In other words, I consider divisor classes of Y corresponding to integer points of $\partial^1\nabla$ (either directly or as one of the irreducible components of the toric divisor's intersection with the Calabi-Yau hypersurface). Such divisors will be called **semitoric**[30].

In order to calculate the intersection number with the generic fiber, one first needs to know the class of the generic fiber. This is particularly simple in the case, where Y a K3 surface[31]. Then, the class of the generic fiber is itself a divisor of Y. The generic fiber is the intersection of the K3 surface Y with the generic fiber of the toric prefibration

$$\begin{array}{ccc} X_{\Delta^{(f)}} & & \\ \downarrow & & \\ X_{\Delta} & \longrightarrow & B, \end{array}$$

i.e. the prefibration fiber over a generic point in the torus $\mathbb{C}^\star$ in the base $B = \mathbb{P}^1$. The generic point in the base is the set of zeroes of the rational function $\chi^{m_b} - c\chi^0$, $c \in \mathbb{C}^\star$ generic, on the base. Its preimage is given by the set of zeroes of the pullback:

$$Z(\chi^{\hat{\pi}^t m_b} - c)$$

Since $\hat{\pi}$ is surjective, $m_f := \hat{\pi}^t m_b$ is one of the two primitive elements of M perpendicular to $\nabla^{(f)}$. The divisor class D_f is then readily calculated as

$$\begin{aligned} D_f &= Z(\chi^{m_f} - c) - div(\chi^{m_f} - c) \\ &= \sum_{i, \langle m_f, n_i\rangle > 0} \langle m_f, n_i\rangle D_i \qquad (6) \\ &= -\sum_{i, \langle m_f, n_i\rangle < 0} \langle m_f, n_i\rangle D_i. \qquad (7) \end{aligned}$$

[30] For Y a K3 manifold, a dense subset of the parameter space of defining polynomials has enhanced Picard group, i.e. $Pic(Y)$ is not generated by irreducible components of the intersections of toric divisors with the K3 surface. For higher dimensional Y and generic (in the algebraic sense) defining polynomials all divisor classes are semitoric.

[31] The following derivation may be easily generalized to higher codimension by using multiple rational functions and intersecting the corresponding divisors.

As usual, D_i are the toric divisors of X_Δ corresponding to primitive ray generators n_i. $div(f)$ denotes the principal divisor of the rational function f. The last equality uses the identity $\sum_i \langle m_f, n_i \rangle D_i = 0$ in $A^1(X_\Delta)$ and reflects the fact, that the result is independent of the choice of m_b.

4.2 Gauge algebras

Given an elliptic K3 surface in terms of a virtual elliptic prefibration $\nabla_{ell} \subset \nabla_{K3}$ and a generic defining polynomial p, we now want to read off the perturbative gauge algebra of a dual heterotic string theory. As explained e.g. in [AKM00] and references therein, this is best done in the type IIA theory compactified on the K3 surface. To obtain F-theory, one first goes to the strong coupling limit opening an effective additional dimension yielding M-theory. Then, the F-theory limit is taken by shrinking all components of the elliptic fiber to zero.

The last step is performed in two stages: First we shrink all components of the fiber not meeting the chosen section of the elliptic fibration[32]. This leaves us with type IIA (or M-theory) on a singular K3 surface. In the second step on shrinks the remaining components to obtain F-theory. The gauge group then already emerges in the type IIA theory[33].

In order to read off the gauge algebra, we thus first have to identify the divisor class of the section. The section is a rational curve, and thus we have to look for effective divisors $\tilde{D}_s$ of self-intersection[34] $\tilde{D}_s^2 = -2$ and intersection $D_f \cdot \tilde{D}_s = 1$ with the fiber.

From the latter condition and (6), (7) we can deduce, that

$$\tilde{D}_s = \tilde{D}_{n_s} + \ldots,$$

where $\tilde{D}_{n_s}$ is an irreducible component of the toric divisor corresponding to a vertex n_s of ∇_{ell} and $(\ldots)$ is an effective semitoric divisor in the orthogonal complement of D_f (i.e. a sum of irreducible components of toric divisors corresponding to points *not* being vertices of ∇_{ell}). Let us first assume that $\tilde{D}_s = \tilde{D}_{n_s}$. Later it will be shown, that this assumption does not restrict generality.

By explicit calculation it can easily be seen, that $\tilde{D}_s^2 = -2$ is already implied by $D_f \cdot \tilde{D}_s = 1$:

If n_s lies in the interior of an edge of ∇_{K3}, all irreducible components of $D_s \cap K3$ automatically have self-intersection -2. This in particular treats the case, where $D_s \cap K3$ splits into multiple irreducible components. In this case it is also clear, that n_s has precisely two neighboring integer points on the common edge, which will be denoted by n_u and n_d. The corresponding semitoric devisors having nonzero intersection with $\tilde{D}_s$ will be denoted $\tilde{D}_u$ and $\tilde{D}_d$. From (6), (7)

[32]Taking the F-theory limit requires a section of the fibration or a B-field in the base [BPS99, BKMT99]. We will not discuss the latter alternative.

[33]in the form of massless twobranes wrapping the singularities. The gauge fields on these branes interact according to the intersection pattern of the exceptional divisors.

[34]Recall, that the adjunction formula forces any curve c algebraically embedded into a surface to have self-intersection $c^2 = 2(g-1)$, where g is the genus of the curve.

and intersection number 1 with the fiber $\tilde{D}_f := D_f \cap K3$ ($D_f \cdot \tilde{D}_s = 1$) we conclude $\tilde{D}_u \cdot \tilde{D}_s = \tilde{D}_d \cdot \tilde{D}_s = 1$.

Now let n_s be a vertex of ∇_{K3}. Then[35], $D_s \cap K3 = \tilde{D}_s$. Again using (6), (7) and $D_f \cdot \tilde{D}_s = 1$ we can conclude that as before n_s must have exactly one neighbor n_u above and n_d below ∇_{ell} on a common edge. In addition, the dual edges to these edges must have integer length 1 and both n_u and n_d must have integer distance 1 from the plane of ∇_{ell} (as measured by m_f).

Denote by e_1 and e_2 the integer points neighboring n_s in ∇_{ell} (which might either lie on a surface or an edge of ∇_{K3}). Both $\{n_s, e_1\}$ and $\{n_s, e_2\}$ must be bases of $N_{ell} \cong \mathbb{Z}^2$. By choice of base we can thus write

$$n_s = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right),\ e_1 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right),\ e_2 = \left(\begin{smallmatrix}\alpha\\-1\\0\end{smallmatrix}\right),\ n_d = \left(\begin{smallmatrix}1\\0\\-1\end{smallmatrix}\right),\ n_u = \left(\begin{smallmatrix}1-\beta\\\gamma\\1\end{smallmatrix}\right).$$

One calculates the (not necessarily pairwise different) face normals

$$m_d^1 = \left(\begin{smallmatrix}-1\\-1\\0\end{smallmatrix}\right),\ m_d^2 = \left(\begin{smallmatrix}-1\\1-\alpha\\0\end{smallmatrix}\right),\ m_u^1 = \left(\begin{smallmatrix}-1\\-1\\\gamma-\beta\end{smallmatrix}\right) \text{ and } m_u^2 = \left(\begin{smallmatrix}-1\\1-\alpha\\\gamma(\alpha-1)-\beta\end{smallmatrix}\right).$$

Since n_s is a vertex of ∇_{ell}, we have $\alpha \leq 1$. The length of the edge θ_d dual to $\overline{n_s n_d}$ is

$$l(\theta_d) = |m_d^1 - m_d^2| = |2-\alpha| \stackrel{!}{=} 1 \qquad \Rightarrow \qquad \alpha = 1,$$

which also ensures $l(\theta_u) = 1$. We must have $\langle m_d^{1/2}, n_u \rangle \geq -1$ and hence $\beta \geq \gamma$ and $\beta \geq 0$.

Denote by E_1, E_2, D_d, D_u the intersections of the K3 surface with the toric divisors corresponding to e_1, e_2, n_d, n_u. For the self-intersection of $\tilde{D}_s$ we obtain

$$\begin{aligned}
\tilde{D}_s^2 &= \langle m_d^2, e_1\rangle \tilde{D}_s E_1 + \langle m_d^2, e_2\rangle \tilde{D}_s E_2 + \langle m_d^2, n_d\rangle \tilde{D}_s D_d + \langle m_d^2, n_u\rangle \tilde{D}_s D_u \\
&= \langle m_d^2, e_1\rangle |m_u^1 - m_d^1| + \langle m_d^2, e_2\rangle |m_u^2 - m_d^2| + \langle m_d^2, n_d\rangle + \langle m_d^2, n_u\rangle \\
&= 0 - 1\cdot\beta - 1 + (\beta - 1) = -2.
\end{aligned}$$

The divisor $\tilde{D}_f$ of the fiber is the class of an elliptic curve. It thus has self-intersection $\tilde{D}_f^2 = 0$, which can also be read off using both (6) and (7). Therefore, the system $\tilde{D}_f$ and $\tilde{D}_s$ span a hyperbolic lattice $H = \langle \tilde{D}_f, \tilde{D}_s + \tilde{D}_f\rangle_{\mathbb{Z}}$. Since H is self-dual, the Picard lattice splits as

$$Pic(K3) = H \oplus H^\perp.$$

This follows from the corresponding splitting of $H^2(K3,\mathbb{Z})$ [Nik80, §1][36]. $H^\perp \subset Pic(K3)$ must be negative definite, because it is contained in the intersection of $H^2(K3,\mathbb{Z})$ with the orthogonal complement of the positive definite threeplane in $H^2(K3,\mathbb{R})$ spanned by the real and imaginary parts of the complex structure and $\tilde{D}_s + 2\tilde{D}_f$.

[35] The following calculations are valid whenever this weaker condition is fulfilled.

[36] Since the reference is not available online, you might want to look up the relevant lemma in [Roh04, Lemma 4.6].

Remark 4.1 *We can now see, why our initial assumption $\tilde{D}_s = \tilde{D}_{n_s}$ does not mean a loss of generality. Any other choice would have been of the form*

$$\tilde{D}'_s = \tilde{D}_s + \alpha \tilde{D}_f + h, \quad \alpha \in \mathbb{Z}, h \in H^\perp.$$

Now ${\tilde{D}'_s}{}^2 = -2 \Rightarrow \alpha = -h^2$. Let $\{h_i; i \in I\}$ be a basis of $H^\perp$. The change of basis $\tilde{D}_s \mapsto \tilde{D}'_s$, $\tilde{D}_f \mapsto \tilde{D}_f$ and $h_i \mapsto h'_i := h_i - (h \cdot h_i)\tilde{D}_f$ is an automorphism of $Pic(K3)$. As $Pic(K3)$ is a primitive sublattice of $H^2(K3, \mathbb{Z})$, one can use the results of [Nik80] to lift this automorphism to $H^2(K3, \mathbb{Z})$ extending the identity on $Pic(K3)^\perp \subset H^2(K3, \mathbb{Z})$. Due to the results of [Roh04], $Pic(K3)^\perp$ always contains a hyperbolic sublattice denoted by $\tilde{H}$ in [Roh04]. If necessary, the lifted automorphism can thus be made orientation-preserving by exchanging the generators of $\tilde{H}$. Being orientation-preserving, the automorphism is induced by a diffeomorphism of the K3 surface [Mat85, Bor86, Don90].

By similar arguments, any two choices of hyperbolic sublattice are connected by a diffeomorphism of the K3 surface. The important point is that in our case the plane of the complex structure (resp. the Picard lattice) is preserved. Note, that one can *have inequivalent (i.e. different modulo automorphisms of $Pic(K3)$) embeddings $H \hookrightarrow Pic(K3)$ when choosing a different n_s.*

Two sublattices of $H^\perp \subset Pic(K3)$ can be easily extracted, namely the lattices Γ_u and Γ_d spanned by semitoric hypersurface divisors corresponding to points above (below) ∇_{ell} apart from $\tilde{D}_u$ ($\tilde{D}_d$).

For dimensionality reasons the space of relations between semitoric divisors is precisely the space of relations between the corresponding divisors of the ambient toric variety, which is just $M \cong M_{ell} \oplus \mathbb{Z}m_f$. $m \in M$ corresponds to the relation

$$\begin{aligned} 0 &\equiv \sum_{n \in \partial\nabla_{K3} \cap N} \langle m, n \rangle D_n \cap K3 \\ &= \sum_{n \in \partial\nabla_{K3} \cap N} \langle m, n \rangle \sum \tilde{D}_n. \end{aligned}$$

The only relations involving only divisors corresponding to points outside the elliptic plane are those given by multiples of m_f and thus always contain $\tilde{D}_u$ and $\tilde{D}_d$ with nonzero coefficients.

Hence, the above semitoric divisors are a base of $\Gamma_u \oplus \Gamma_d$. By negative definiteness they must be rational curves.

So far, I have only talked about smooth elliptically fibered K3 surfaces. We now shrink to zero all the rational curves in $\Gamma_u \oplus \Gamma_d$. If there is anything to shrink, our K3 surface will develop (one or) two pointlike singularities located at the points 0 and ∞ over the base $\mathbb{P}^1$. The ADE classification of these singularities is then nothing else than the intersection form on exceptional divisors of the blowup (i.e. the original smooth K3 surface) and hence coded in the intersection matrix on Γ_u and Γ_d, respectively. By determining the ADE type, one also reads off the gauge algebras corresponding to these singularities[37].

[37]The intersection form is minus the Cartan matrix, i.e. (all roots have equal length) is

Remark 4.2 *If none of the toric divisors corresponding to points in the upper and lower half of the K3 polyhedron splits into multiple semitoric divisors, the Dynkin diagrams may be read off directly from the polyhedron's edge diagram as observed in [CF98]. The only detail not directly obvious from the edge diagram is that divisors corresponding to vertices of ∇_{K3} have the correct self-intersections, which follows from negative definiteness.*

In the split case, one can still read off the Dynkin diagram, if one splits the points and edges corresponding to split toric divisors and the intersections between them. Remember, that in contrast to the nonsplit case, the points above and below the section point must not be omitted, but only one irreducible component of each of them.

Remark 4.3 *In simple examples (as those mainly studied in [PS97]), one has $\Gamma_u \oplus \Gamma_d = H^\perp \subset Pic(K3)$. This, of course, is not always the case. In particular, two things can happen:*

i) There could be rational curves apart from those in $\Gamma_u \oplus \Gamma_d$ which one can shrink to zero to obtain enhanced gauge symmetry. Shrinking to zero divisors corresponding to vertices of ∇_{ell} leads to a globally singular "elliptic" fibration, since these divisors do meet the generic fiber. Divisors corresponding to points on edges of both ∇_{ell} and ∇_{K3}, on the other hand, do not meet the generic fiber. Since they do *meet the generic K3 hypersurface, shrinking them to zero introduces singularities over points in $\mathbb{T} \subset \mathbb{P}^1$.*

Rational curves corrsponding to such points intersect neither those in Γ_u and Γ_d nor those corresponding to points on different edges of ∇_{ell}. Hence, one might only obtain additional gauge group factors of type A_n.

ii) The lattice $\Gamma_u \oplus \Gamma_d$ is not primitive. This happens, whenever the integer points in the elliptic plane not lying on hypersurfaces of ∇_{K3} together with n_u and n_d do not generate N: Precisely in this case there is a primitive relation $m \in M$ between a nonprimitive element of the lattice spanned by the divisors corresponding to the above points and an element of the obvious lift of $\Gamma_u \oplus \Gamma_d$.

The occurence of additional A_n factors as described in (i) requires a common edge of ∇_{ell} and ∇_{K3}. The integer points lying on this edge provide a base for N_{ell}. Therefore, both phenomena cannot occur simultaneously.

This is not a problem, but rather an interesting feature. It is related to the fact, that $\Gamma_u \oplus \Gamma_d$ are the root lattice of the corresponding gauge algebra, which often only is a sublattice of the weight lattice. To illustrate the point, consider the "heterotic duality" polyhedron (number 4319 in [Rohb]) with vertices

$$n_{E_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ n_F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ n_{C_0} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} \text{ and } n_{C_{12}} = \begin{pmatrix} -9 \\ -8 \\ -6 \end{pmatrix}.$$

proportional to the Killing form. The latter statement directly extends to $\Gamma_u \oplus \Gamma_d \oplus H$: Here the intersection form is an invariant bilinear form on the Cartan subalgebra of the corresponding extended untwisted affine Kac-Moody algebra.

This polyhedron allows two elliptic fibrations (numbers 13277 and 13278 in [Rohb]) with section corresponding to the planes with normals

$$\tilde{m}_f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } m_f = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

While the former leads to gauge group $E_8 \times E_8$, the latter leads to gauge algebra D_{16}.

Apart from the vertices, the integer points on edges of ∇_{K3} are

$$n_S = \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix},\ n_d = \begin{pmatrix} -6 \\ -5 \\ -4 \end{pmatrix},\ n_{E_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},\ n_{A_1} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix},\ n_{A_2} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix},$$

$$n_{C_{13}} = \begin{pmatrix} -4 \\ -4 \\ -3 \end{pmatrix} \quad \text{and} \quad n_{C_i} = \begin{pmatrix} 3-i \\ 4-i \\ 6-i \end{pmatrix}, i = 1\ldots, 11.$$

n_{E_1}, n_{E_2} and n_S are the vertices of the fiber polyhedron. The divisor S is a section (so is, by symmetry, E_2) and, together with F, spans our hyperbolic lattice H. With notations as before, $n_u = n_F$ and $\Gamma_u = \{0\}$. A base for Γ_d is given by $A_1, A_2, C_0, \ldots C_{13}$. The corresponding points, together with the edge segments between them, form the Dynkin diagram of D_{16}. Simply counting dimensions, one might think that $Pic(K3) = H \oplus \Gamma_d$. This is not true, because $n_{E_1}, n_{E_2}, n_u, n_d$ and n_S only generate an index two sublattice of N. Thus, one needs one additional generator for $H^\perp$. A simple calculation evaluating the relations in M shows that one can take

$$E_1 - 2E_2 \equiv \frac{1}{2}\left(5A_1 + 8A_2 + \sum_{i=0}^{13}(12-i)C_i\right),$$

which completes $H^\perp$ to the weight lattice of $spin(32)/\mathbb{Z}_2$.

Note, that one can also derive the qualitative result without doing any calculations at all: Considering the fibration given by $\tilde{m}_f$ with gauge group $E_8 \times E_8$, one knows that $Pic(K3)$ must be self-dual and so must then be our $H^\perp$. Since there are only two even self-dual negative definite lattices in 16 dimensions, it must be the weight lattice of $spin(32)/\mathbb{Z}_2$.

iii) If one does not wish to ignore additional gauge algebra summands as discussed in (i), the additional summand can easily be calculated. The only thing one has to remember is to omit semitoric divisors corresponding to points on common edges of ∇_{K3} and ∇_{ell}, which are neighbors of the chosen section. For the additional A_n summands, a situation as described before for nontoric sections (omitting only one irreducible component of a toric divisor) cannot occur: For nontoric section n_s must lie on a common split edge of ∇_{K3} with n_u and n_d and would thus need to be a vertex in order to also lie on a common edge of ∇_{K3} and ∇_{ell}. This, in turn, contradicts the assumption of the section to be nontoric. Arguments similar to the discussion of $\Gamma_u \oplus \Gamma_d$ show, that one indeed obtains a sublattice $\Gamma_u \oplus \Gamma_d \oplus A$, where A denotes the additional gauge algebra summand as

directly read off from the polyhedron. In (ii) I argued, that $\Gamma_u \oplus \Gamma_d$ must be a primitive sublattice of $H^\perp$. This is not *true for the full sum $\Gamma_u \oplus \Gamma_d \oplus A$. Primitivity can fail if we cannot find a base for N_{ell} consisting of vertices of N_{ell} only.*

Elliptically fibered toric K3 surfaces (overview)	
reflexive polyhedra	4,319
elliptic fibrations	13,278
with section	12,060
nontoric section only	729
nonabelian gauge algebra	11,890
non-simply laceable	4,505
nontoric section	376

Table 1: **Elliptically fibered toric K3 surfaces.** *For elliptically fibered K3 surfaces, all virtual elliptic fibrations are fibrations. The numbers in the table refer to virtual elliptic fibrations after identification via automorphisms of the K3 polyhedron. "Non-simply laceable" means, that non-simply laced gauge algebras might emerge when the elliptic K3 is the generic fiber of an elliptic K3 fibration (c.f. section 4.3).*

I have searched all 4319 reflexive polyhedra from [KS98b] corresponding to toric K3 hypersurfaces for toric elliptic fibrations. For all fibration structures, I checked for toric and nontoric sections and the corresponding gauge algebras by calculation and classification of the intersection matrices of Γ_u and Γ_d. Lists containing the results can be found at [Rohb]. An overview is given in table 1.

4.3 Monodromy

The main idea for extending the $Het \leftrightarrow F$ duality to higher dimension is to use elliptic K3 fibrations and fiberwise duality. If one demands purely geometric phases, the F-theory compactification space in fact must be a K3 fibration [AL96]. A phenomenon typical to fibrations can modify the gauge group emerging in the pure K3 case, namely monodromy. Following a closed path in the base of the K3 fibration induces an automorphism of the K3 fiber. This induces an automorphism of the Picard lattice, which does not need to be trivial. The gauge group then only emerges as the monodromy invariant part of the original Lie algebra. The relevant automorphisms of Lie algebras are depicted in figure 1.

When studying toric eliptic K3 fibrations, one is interested in a simple criterion for the occurrence of such identifications. As the assumptions made in [PS97] are not always valid, the simple criterion given there using only properties of ∇_{K3} will not suffice. We will see, though, that it can easily be extended to full generality by taking into account simple properties of the polyhedron

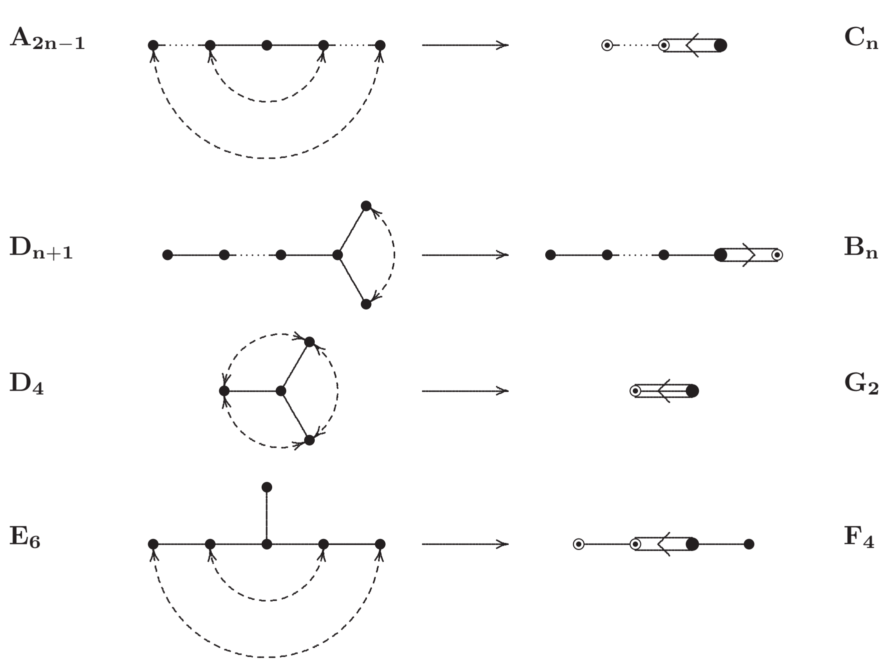

Figure 1: **Outer automorphisms of Lie algebras.** *The dashed arrows connect roots interchanged by an outer automorphism. Note, that Cartan matrix and intersection form are related by a simple switch of sign only when all roots have equal length (i.e. on the left hand side). The direction of the arrows on the right hand side may easily be deduced by using Serre relations on Chevalley generators of the invariant subalgebra. The problematic case of outer automorphisms of A_{2n} [AKM00] cannot occur, because a partial split toric edge diagram can only be connected when including a vertex point (which is nonsplit).*

∇_{CY} corresponding to our elliptic K3-fibered toric Calabi-Yau hypersurface. Consider a toric K3 fibration

$$\begin{array}{ccc} K3 \subset X_{\Delta_{K3}} & & \\ \downarrow & & \\ CY \subset X_{\Delta_{CY}} & \longrightarrow & B \end{array}$$

with virtual prefibration $\nabla_{K3} \subset \nabla_{CY}$.

The toric divisors of the generic prefibration fiber are just the intersections of the fiber with the toric divisors of $X_{\Delta_{CY}}$ corresponding to integer points in the K3 subpolyhedron ∇_{K3}. Hence, they cannot be interchanged when moving around in the prefibration base. Obviously, the same argument prevents toric divisors in the K3 fiber from being permuted. We thus only have to find

out, whether the nontoric irreducible components of a given toric divisor are interchanged by monodromy. First assume (as was done in [PS97]) that the corresponding divisor of $X_{\Delta_{CY}}$ is irreducible. Assume furthermore, that the irreducible components of the fiber divisor split into multiple orbits of the monodromy group acting on them. For each orbit, we obtain an irreducible component of the divisor of $X_{\Delta_{CY}}$ by moving around an irreducible component of the fiber divisor over the base. We thus obtain multiple irreducible components of the divisor of $X_{\Delta_{CY}}$ in contradiction to the assumption.

Now assume, that the toric divisor of $X_{\Delta_{CY}}$ also splits into multiple irreducible components. We will see, that no identifications by monodromy occur in this case. Denote by $n_d \in \nabla_{K3} \subset \nabla_{CY}$ the point corresponding to our reducible toric divisor, which must lie on an edge $\theta^\star_{K3}$ of ∇_{K3} and in the interior of a codimension 2 face $\theta^\star_{CY} \supset \theta^\star_{K3}$ of ∇_{CY}. Then the preimage of the dual edge θ_{K3} of Δ_{K3} under π_M must precisely be the edge θ_{CY} of Δ_{CY} dual to $\theta^\star_{CY}$. Hence, the toric divisor of the K3 fiber splits into exactly equally many irreducible components as the divisor of the total Calabi-Yau variety, showing that no identifications occur.

The same argument applies to the section. A given nontoric section of the generic elliptic K3 fiber of a toric elliptic K3 fibration patches together to yield a section (in contrast to a multisection) of the total elliptic fibration precisely when the corresponding point lies in the interior of a codimension two face of ∇_{CY}.

4.4 A Note on Generality

One may wonder whether we have lost elliptic fibrations by requiring them to be toric, i.e. to be induced by a toric prefibration. This is most certainly the case. A more general approach would be to search for possible classes of the generic fiber, i.e. to search for a pencil of effective divisors with vanishing self-intersection. Finding such divisors is technically difficult since the equation one wants to solve is quadratic rather than linear.

In many cases without toric elliptic fibration, one can at least show that a more general elliptic fibration does not exists, either. This is clear for the cases with Picard number 1. In these cases the unique divisor class has positive self-intersection. Even when we have divisor classes with both positive and negative self-intersection, often no divisor class with vanishing self-intersection exists. For an easy example, consider the reflexive polyhedron (no. 8 in the list at [Rohb]) with vertices

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

The generic hypersurface has Picard number 2 and using the formulae in appendix B one calculates the intersection matrix to be

$$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$$

I make the ansatz $E = xD_1 + yD_2$ for the class of an elliptic fiber with $x, y \in \mathbb{Z}$. One then calculates

$$-2x^2 + 2y^2 + 2xy \stackrel{!}{=} 0 \quad \Rightarrow \quad x_{1,2} = y\frac{1 \pm \sqrt{5}}{2},$$

which has no solution in $\mathbb{Q}^2$.

5 Fourfolds

As stated before, one obtains models with four flat space-time dimensions by compactification of the heterotic string on a Calabi-Yau threefold with vector bundle. F-theory, on the other hand, has to be compactified on an elliptic Calabi-Yau fourfold.

For an overview of different ways to obtain explicit examples of Calabi-Yau fourfolds as well as general properties and relations between invariants of such varieties, the interested reader is referred to [KLRY98]. In order to use the methods presented in the preceding section, the focus of interest in this paper is naturally given by hypersurfaces in five-dimensional toric varieties.

Due to the lack of a complete classification of reflexive Polyhedra in five dimensions one has to resort to constructible subclasses of polyhedra in order to obtain explicit examples to study. With the focus of obtaining families of varieties well adopted to the study of singular transitions a large class of toric Calabi-Yau fourfolds was constructed in [KLRY98]. These fourfolds are constructed as fibrations and thus one might miss interesting peculiarities of fibration structures when restricting attention to such a class of objects.

Hence, I decided to base my studies on a class of fourfolds obtained independently of any fibration structures: the class of transverse hypersurfaces in weighted projective spaces.

5.1 Hypersurfaces in weighted projective spaces

In all dimensions, weighted projective spaces are toric varieties. Nevertheless, the relationship between transverse hypersurfaces in weighted projective spaces and toric Calabi-Yau hypersurfaces in the sense used in this paper is not a direct one. This is due to the fact, that the fans of weighted projective spaces are always fans over the faces of simplices, but in general these simplices are not reflexive.

The reflexive polytope related to a weighted projective space is rather given by its maximal Newton polyhedron, i.e. the convex hull of the homogenous monomials of the appropriate degree to define a Calabi-Yau hypersurface.

In dimensions less or equal to four, all Newton polyhedra obtained in this way are reflexive. Though the original weighted projective spaces are too singular to be members of the corresponding families of toric varieties[38], suitable blow-ups

[38] In particular, generic hypersurfaces can fail to be Σ-regular with respect to the fan of the weighted projective space.

are.

In contrast, in dimensions bigger or equal to five, not all maximal Newton polyhedra are reflexive, i.e. suitable blow-ups do not exist. In fact, in five dimensions only about a fifth of all weight sets allowing transverse polynomials give rise to reflexive polyhedra.

There are 1.100.055 sets of weights[39] allowing for transverse hypersurfaces. They were calculated in [LSW99] using the algorithm of [KS94] used to classify transverse threefold hypersurfaces.

Determining the reflexive polyhedra is a conceptually trivial task. One simply calculates the Newton polyhedra by enumerating monomials, moves the inner point (all exponents 1) to 0, makes a change of base to the orthogonal complement of the weight vector and calculates the convex hull of the resulting point set. Expressing the convex hull in terms of its bounding halfspaces makes it a trivial task to check for reflexivity. The only problem is posed by the huge amount of data. I calculated the convex hulls[40] and in case of reflexivity also the Hodge numbers by application of the formulae in [Bat94, KLRY98]. In all cases, they agree with the numbers found in [LSW99] using Landau-Ginsburg methods.

In addition, the reflexive polyhedra were scanned for equivalences[41] modulo $GL(5,\mathbb{Z})$. Though from a physicist's point of view, Mirror symmetry for fourfolds does not play the same role as for threefolds, I also checked whether the dual polyhedra also arise as Newton polyhedra for transverse weights.

The reflexive Newton polyhedra were independently also calculated by the authors of [KS98a][42].

A short summary of the results is found in table 2. For more details c.f. section 5.5.

5.1.1 Singularities

As already mentioned above, the majority of four-dimensional transverse hypersurfaces in weighted projective spaces are too singular for their maximal Newton polyhedra to be reflexive. The simplest example for this is the (well known) family

$$\mathbb{P}_{1,1,1,1,1,2}[7]$$

of degree 7 hypersurfaces in $\mathbb{P}_{1,1,1,1,1,2}$.

[39] Here, the well known equivalences between sets of weights have of course been modded out. This is already needed to make the set finite.

[40] Since I was also interested in the complete face structure of the polyhedra, I found it (by experiment) useful not to use a beneath-beyond type algorithm, but rather a higher dimensional analogon to the gift-wrapping algorithm. In most cases where I tried both types of algorithm, the latter (highly output-sensitive) algorithm proved to be faster.

[41] One only has to try to find a group element transforming the polyhedra into each other if all known invariants are equal. In particular, I used the Hodge numbers and the individual numbers of faces of different dimensions to perform a preselection.

[42] Their results can be found at `http://hep.itp.tuwien.ac.at/~kreuzer/pub/CY4`.

Everything found in these lists agrees with my own calculations, which is a good test for the correctness of both calculations.

Transverse weights and reflexive polyhedra (overview)				
transverse weights	1.100.055	100 %		
dual Hodge tripel	425.859	38,7 %		
different Hodge triples	667.954	60,7 %	100 %	
reflexive polyhedra	252.933	23,0 %		100 %
different Hodge triples	158.178		23,7 %	62,5 %
different mod. $GL(5,\mathbb{Z})$	202.746			80,2 %
dual Hodge tripel	90.390			35,7 %
dual polyhedron	31.778			12,6 %

Table 2: **Transverse weights and reflexive polyhedra.** *The entries labelled* "different ..." *are lower and upper bounds for the number of inequivalent families of hypersurfaces. Equality of Hodge numbers is a necessary and equivalence of polyhedra a sufficient condition for equality of the corresponding families. Under* "dual ..." *weight sets are counted, where the dual Hodge triple/polyhedron also occurs in the same class of weight sets.*

But even if the Newton polyhedra are reflexive, the fourfolds can have terminal singularities, i.e. singularities which cannot be resolved while preserving the canonical class (i.e. without violating the Calabi-Yau condition). This is related to the fact, that basic[43] simplices are automatically elementary[44] in dimensions 2 and less only. For Calabi-Yau threefolds this is enough to guarantee smoothness of maximal crepant partial desingularizations: the relevant cones have dimension 3 and one additional dimension is "won" by reflexivity, which forces the cones to be index 1 Gorenstein cones[45]. In higher dimensions, this does not work anymore.

For a simple example consider the family

$$\mathbb{P}_{1,1,1,1,2,2}[8].$$

The maximal Newton polyhedron is a reflexive simplex. In coordinates suitable for the purposes of this section its vertices are given by

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Now consider the face containing the four vertices corresponding to the first four columns. Going to the lattice subspace defined by the face, this face is the

[43] i.e. simplices not containing integer points apart from their vertices

[44] i.e. having volume 1 in lattice units

[45] i.e. $\sigma = \langle n_1, \ldots, n_k \rangle_{\mathbb{R}+}$ and there exists a primitive integer vector m such that $\forall i : \langle m, n_i \rangle = 1$.

simplex with vertices

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

This is a basic simplex. Therefore, we cannot subdivide the cone over this face without changing the canonical class and thereby violating the Calabi-Yau-condition for the generic embedded hypersurface. On the other hand, the simplex is *not* elementary — in lattice units it has volume 2. Thus, the four vertices cannot be part of a basis for $\mathbb{Z}^5$ and we have a pointlike singularity of the generic hypersurface (the singular locus in the toric variety is one-dimensional).

In the preceding example, the singularity could not be avoided. The general situation is even more complicated: Different maximal triangulations (even different projective triangulations) can lead to different numbers of terminal singularities. In this way, terminal singularities can occur in phases of models, which also possess a smooth phase. As a local example for this kind of phenomenon, consider the cone[46] $\sigma = \langle a, b, c, d, e \rangle_{\mathbb{R}^+}$ with generators

$$a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ d = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The convex hull of the generators is an octahedron. σ has two different maximal crepant partial desingularizations, which are given by the two maximal triangulations of this octahedron. The two triangulations[47] are depicted in figure 2.

The first possibility is to split σ into the three cones $\sigma_1 = \langle a, b, c, e \rangle_{\mathbb{R}^+}$, $\sigma_2 = \langle a, c, d, e \rangle_{\mathbb{R}^+}$ and $\sigma_3 = \langle a, b, d, e \rangle_{\mathbb{R}^+}$. As the volume of the octahedron is 3, all three resulting open patches are smooth.

The second possibility is to split into the two cones $\sigma_1 = \langle a, b, c, d \rangle_{\mathbb{R}^+}$ and $\sigma_2 = \langle b, c, d, e \rangle_{\mathbb{R}^+}$. While the open patch corresponding to σ_1 is again smooth, σ_2 is (up to a change of base) just the cone from the preceding example and contains a terminal singularity at its distinguished point.

Although terminal singularities occur for generic members of families of four-dimensional Calabi-Yau hypersurfaces in toric varieties, they are of a comparatively mild kind (not only for hypersurfaces in weighted projective spaces):

Theorem 5.1 *The terminal singularities of generic members of families of four-dimensional Calabi-Yau hypersurfaces in toric varieties corresponding to reflexive polyhedra are at most cyclic quotient singularities.*

Proof: The relevant cones for terminal singularities of generic hypersurfaces are four-dimensional simplicial cones. Due to reflexivity these are always index

[46] This cone e.g. occurs in the phase structure of the reflexive hypercube.

[47] Both triangulations are regular, i.e. allow for a strictly convex piecewise linear function. This, in turn, guarantees the existence of a Kähler form on the resulting variety.

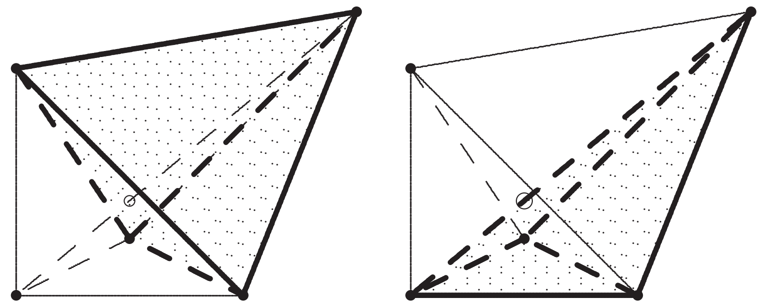

Figure 2: *The two maximal triangulations of the basic octahedron. In both cases, one of the simplices is shaded and its edges are drawn bold. The circular marks in the centers demark the intersection point of the diagonal with the border of the lower left simplex. This point is not integer.*

1 Gorenstein cones and can (by choice of base) be written as

$$\sigma = \left\{ r \cdot \left(\begin{smallmatrix} x \\ 1 \\ 0 \end{smallmatrix} \right) \mid x \in \Delta \subset \mathbb{R}^3, r \in \mathbb{R}^+ \right\},$$

where Δ is a basic but not elementary simplex. All of its facets are two-dimensional and must hence be elementary in their respective lattice subspaces. Therefore one can always choose a base such that

$$\Delta = \text{c.h.}\{ \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right) \}$$

with $a, b, c \in \mathbb{Z}$, $c > 1$. From the holomorphic quotient construction one has

$$U(\sigma) = \mathbb{C}^\star \times \mathbb{C}^4 / G,$$

where $G = \mathrm{Hom}(A, \mathbb{C}^\star)$ and A is determined by the exact sequence

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{\alpha} \mathbb{Z}^4 \xrightarrow{\beta} A \longrightarrow 0$$

with

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ a & b & c & 1 \end{pmatrix}.$$

Hence, $A = \langle \beta((0, 0, 0, 1)) \rangle = \mathbb{Z}_c \Rightarrow G = \mathbb{Z}_c$. $\square$

5.2 Virtual fibrations

The first step for finding toric Calabi-Yau fibrations is identifying virtual fibrations. The objects of most interest are elliptic and K3 fibrations, and thus it

would have been desirable to do a complete scan for both types of virtual fibrations. Unfortunately, a scan for general K3 fibrations would have exceeded my possibilities. A search using sublattice enumeration (c.f. section 3.4.1) forbids itself because of its bad scaling behavior. A search by classification (c.f. section 3.4.2) is less out of reach, but still exceeded my possibilities due to the size of the classification list.

For this reason, I concentrated my search on elliptic and elliptic K3 fibrations. For the elliptic fibrations, the fiber polyhedron can be one of the 16 reflexive polyhedra[48] and a search by classification could be implemented efficiently. The results of this scan can then be extended to virtual elliptic K3 fibrations using sublattice enumerations. Due to the known sublattice given by the two-dimensional subpolyhedron, this extension search now only[49] depends linearly on the number of integer points in ∇_{K3}.

As the expense of checking virtual fibrations for obstructions (c.f. section 5.3 below) also strongly depends on the number of points in ∇_{CY}, I still had to reduce my program to a subset of all polyhedra identified in section 5.1. As a compromise between maximal exhaustiveness and minimal computational expense for the calculations on a single polyhedron ∇_{CY}, I arbitrarily restricted to polyhedra ∇_{CY} with no more than $10,000$ integer points. The restricted set consists of $222,653$ polyhedra, which is roughly 88 % of the 252,933 polyhedra corresponding to hypersurfaces in weighted projective spaces.

5.3 Obstructions for fibrations

Assume we have found a virtual toric elliptic K3 fibration

$$\nabla_{ell} \subset \nabla_{K3} \subset \nabla.$$

How can we find out, whether this gives in fact rise to a fibration? As noted before, the simple approach of trying all possible triangulations of $\partial^1\nabla$ is computationally much too expensive except for the simplest cases.

Hence, we are looking for a simple way to construct a fan Σ_b and a compatible triangulation of $\partial^1\nabla$ in a way that excludes their existence in the case of failure. Where necessary, we will pose mild additional conditions on the fibrations, i.e. on Σ_b and the triangulation.

5.4 Constructing projective triangulations

The general approach in the construction of compatible fans Σ and Σ_b will be the following: We first construct the coarsest fan Σ_b meeting a given set of conditions and then subdivide the fan over codimension ≥ 2 faces of ∇ to obtain a fan Σ compatible with Σ_b. At times, we will further subdivide Σ_b, which then

[48] Mainly for reference reasons in other lists, a list of these polyhedra may be found at [Rohb].

[49] It obviously also depends on the number of elliptic subpolyhedra $\nabla_{ell} \subset \nabla_{CY}$, which is small in most cases.

forces us to further subdivide Σ. In the end, any simplicial refinement of Σ will be compatible with Σ_b.

Since we want to obtain a projective fan Σ, we must ensure, that we do not accidentally introduce obstructions against the existence of stricly convex piecewise linear functions in the process of subdividing (the normal fan of Δ allows such functions).

It is well known, that star-subdividing preserves projectiveness[50]. Unfortunately, this is not what we will be doing. I will therefore present some additional types of subdivision, which preserve projectiveness. We will need the following generalization of a fan:

Definition 5.2 *A collection of (not necessarily strictly) convex rational polyhedral cones is called a* ***quasifan****, if the following holds:*

i) If $\sigma \in \Sigma$ and $\tau \prec \sigma$ is a facet of σ, then $\tau \in \Sigma$.

ii) If $\sigma, \sigma' \in \Sigma$ and $\tau := \sigma \cap \sigma'$, then $\tau \prec \sigma$ and $\tau \prec \sigma'$.

The only missing ingredient in comparison with a fan is, that we do not demand the cones to be strictly convex. Obviously, one can intersect quasifans to obtain new quasifans just as one can do with fans. The intersection of a fan with a quasifan is a fan. As with fans, a quasifan is called projective, if it allows a strictly convex piecewise linear function.

We need the following almost obvious facts:

Lemma 5.3 *Let Σ, Σ' be two finite projective quasifans. Then their intersection is a projective quasifan.*

Proof: The sum of two convex functions is convex. After scaling one of them (if necessary) the sums of linear pieces differ whenever the pieces of one summand differ. $\square$

Lemma 5.4 *Let $\pi : \mathbb{Q}^n \twoheadrightarrow \mathbb{Q}^m$ be a linear map and Σ a regular quasifan with support in $\mathbb{Q}^m$. Then $\Sigma' = \{\pi^{-1}\sigma; \sigma \in \Sigma\}$ is a projective quasifan.*

Proof: A piecewise linear strictly convex function Ψ on Σ yields the strictly convex function $\Psi \circ \pi$ on Σ'. $\square$

5.4.1 Codimension 2: K3 fibrations

While finding virtual fibrations is easiest for small dimension of the fiber, checking for obstructions is easiest for small codimension of the fiber. We have seen before, that in the case of codimension 1 of the fiber, no obstructions can occur.

[50] All cones containing the ray ρ around which one star-subdivides are of the form $\tilde{\sigma} = \sigma + \rho$, where σ is a cone in the original fan with $\dim \tilde{\sigma} = \dim \sigma + 1$. Hence, the linear functions on these cones given by a strictly convex function on the old fan can be consistently deformed by changing the value on a generator of ρ to provide a strictly convex function on the subdivided fan.

Unfortunately, the minimal fiber codimension in our setting is two, which is the case I will discuss first.

Here, the fan of the toric base variety lives in the plane, where a complete fan is uniquely determined by its rays. In addition, any two-dimensional fan is projective.

I will first discuss triangulations compatible with a flat prefibration, i.e. a prefibration with pure dimensional fibers[51]. One easily sees by considering restrictions to torus orbits[52], that this is equivalent to the requirement, that any cone in Σ maps onto a cone in Σ_b. In particular, the image of any ray in Σ under $\hat{\pi}$ must be either a ray of Σ_b or the origin. If we want to find a maximal triangulation of $\partial^1\nabla$ corresponding to a maximal crepant desingularization of the total Calabi-Yau variety, our base fan is now already completely fixed.

It remains to check, whether a complete compatible triangulation of $\partial^1\nabla$ exists. Our fan Σ (if it exists) will be a refinement of the fan $\tilde{\Sigma}$ over codimension ≥ 2 faces of ∇. In order to be compatible with Σ_b, it must be a refinement of the fan $\tilde{\Sigma}'$ obtained by intersecting the cones in $\tilde{\Sigma}$ with the quasifan given by projection preimages of cones in Σ_b. Since Σ_b is projective so is $\tilde{\Sigma}$ due to lemma 5.3 and 5.4.

For Σ to exist, the cones in $\tilde{\Sigma}'$ must be generated by integer elements of $\partial^1\nabla$. If they are, any complete projective triangulation refining $\tilde{\Sigma}'$ will be compatible. With lemma 5.5 below it suffices to check intersections with preimages of rays of Σ_b.

Lemma 5.5 *Let $A \in \mathbb{Q}^n$ be an affine hyperplane, $S \subset A$ a finite set and $\sigma \subset \mathbb{Q}^n$ an n-dimensional strongly convex polyhedral cone generated by a subset of S. Let $\pi : \mathbb{Q}^n \to \mathbb{Q}^2$ be a linear map and $\sigma_2 \subset \mathbb{Q}^2$ a strongly convex cone. Set $\sigma^\cap := \sigma \cap \pi^{-1}\sigma_2$. The following are true:*

(i) Let $\dim \sigma_2 = 2$. Then $\sigma^\cap$ is generated by elements of S if and only if the intersections $\sigma \cap \pi^{-1}\tau_1$ and $\sigma \cap \pi^{-1}\tau_2$ are generated by elements of S, where τ_1 and τ_2 are the two facets (bounding rays) of σ_2.

(ii) Let $n > 2$ and $\dim \sigma_2 = 1$. Then $\sigma^\cap$ is generated by elements of S if and only if $\sigma \cap \pi^{-1}(0)$ and the intersections $\tau \cap \pi^{-1}\sigma_2$ are generated by elements of S for all proper faces $\tau \prec \sigma$.

Proof: $\sigma^\cap$ is a strongly convex rational polyhedral cone. A minimal set of generators must consist of generators of dimension 1 faces by convexity. In both

[51] For generic defining polynomial, the fibration fibers will also be pure dimensional.

[52] This works as follows: The torus orbits are in 1:1 correspondence to the cones of the fan, the dimension of an orbit O_σ being $\operatorname{codim}\sigma$. Due to covariance of the torus actions, the dimension of the fiber over a point in a torus orbit O_{σ_b} of the base is the largest difference between the dimension of a preimage orbit O_σ and the dimension of O_{σ_b}. An orbit $O_\sigma, \sigma \in \Sigma$, is mapped onto the orbit O_{σ_b} corresponding to the smallest cone σ_b with $\pi(\sigma) \subset \sigma_b$. For a flat fibration the dimension of the fiber must not exceed that of the generic fiber (the other direction is ensured by lemma 3.1). Hence, $\dim O_{\sigma_b} + \dim N - \dim N_b \geq \dim O_\sigma \Leftrightarrow \dim \sigma \geq \dim \sigma'$. This means, that no cone $\sigma \in \Sigma$ with $\dim \sigma < \dim N_b$ may be mapped into the interior of a higher dimensional cone. Together with lemma 3.1 this yields the assertion.

cases, one direction follows from the fact, that all faces of $\sigma^\cap$ are generated by subsets of a minimal set of generators for $\sigma^\cap$. In other words: If $\sigma^\cap$ is generated by a subset of S, so are all of its faces.

For the other directions, first consider case (i): If an element of a minimal set of generators of σ is not contained in either $\sigma \cap \pi^{-1}\tau_1$ or $\sigma \cap \pi^{-1}\tau_2$, it must already generate a dimension 1 face of σ and hence is a multiple of an element of S.

For case (ii), assume a minimal set of generators for $\sigma^\cap$ contains a vector ρ in the interior of σ and $\pi(\rho) \neq 0$. Since $n > 2$, we can find $0 \neq x \in \mathbb{Q}^n : \pi(x) = 0$. Consider the line $\rho(t) := \rho + tx$, which is completely contained in $\pi^{-1}\sigma_2$. Since ρ is in the interior of σ, an open interval around 0 is also contained in $\sigma^\cap$. Since ρ and x must be linearly independent, this contradicts the assumption that ρ generates a dimension 1 face of $\sigma^\cap$. $\square$

Restricting to subspaces containing faces of σ and using complete induction on n, we obtain:

Corollary 5.6 *Let A, S, σ and σ_2 be as in lemma 5.5(ii). $\sigma^\cap$ is generated by elements of S, if and only if $\sigma \cap \pi^{-1}(0)$ and the intersections $\tau \cap \pi^{-1}\sigma_2$ are generated by elements of S for all proper faces $\tau \prec \sigma$ with* $\dim \tau \leq 2$.

More general fibrations can only be obtained by omitting rays in our preliminary Σ_b. Compatible triangulations can obviously only be constructed when omitting all rays in Σ_b for which the above subcones failed to be generated by integer elements of $\partial^1\nabla$. But if the remaining rays still define a complete fan, i.e. if there are at least two remaining rays in any halfspace of N_b, compatible triangulations are constructible just as before.

For practical reasons and use in the sequel I will still define another condition on the prefibration: If the images of the rays over all vertices of ∇ are contained in the fan Σ_b, the fibration will be called **singularly flat**. The name is due to the fact, that such a fibration becomes flat at some boundary of the family of maximally crepantly desingularized hypersurfaces corresponding to Δ.

5.4.2 Codimension 3: Elliptic fibrations

For an elliptic fibration of a Calabi-Yau fourfold the codimension of the generic fiber and hence the dimension of the base is three. Thus, the base fan is no longer determined by its rays alone. In order to use methods similar to the two-dimensional case, we would have to enumerate the possible sphere triangulations corresponding to a given set of rays. This is not as bad as enumerating all possible triangulations of $\partial^1\nabla$, but in many cases still very expensive. Even worse, in three dimensions cones are not forced to be simplicial and for a complete enumeration of possible base fans we must also consider fans obtained by omitting lines from a triangulation.

Fortunately, our main interest is not in elliptic fibrations, but in elliptic K3

fibrations. Here we can use the fact, that the base fibration

$$\begin{array}{ccc} \mathbb{P}^1 & & \\ \downarrow & & \\ X_{\Sigma_{b,e}} & \xrightarrow{\pi_2} & X_{\Sigma_{b,K}} \end{array}$$

puts severe restrictions on the possible fans $\Sigma_{b,e}$ compatible with a given fan $\Sigma_{b,K}$: All cones in $\Sigma_{b,e}$ must project into cones of $\Sigma_{b,K}$.

$\Sigma_{b,e}$ is a complete fan. Writing $N_{b,e} = \mathbb{Z} \oplus N_{b,K}$, it must thus contain the rays generated by $(1,0,0)$ and $(-1,0,0)$, which constitute the subfan corresponding to the fiber $\mathbb{P}^1$. This poses no constraints on the virtual fibrations, because the preimages of these rays are just the upper and lower halves of N_{K3} with respect to N_{ell}. Now depict the metric unit sphere in $N_{b,e}$ with north and south pole defined by the above two rays and subdivided by the meridians defined by the rays of $\Sigma_{b,K}$ (they are the intersections of the sphere with the preimages of the rays under $\hat{\pi}_2 : N_{b,e} \to N_{b,K}$). Any cone of $\Sigma_{b,e}$ is uniquely determined by its intersection with the sphere. In order to be compatible with the K3 fibration, none of the sphere segments corresponding to cones in $\Sigma_{b,e}$ is allowed to cross the meridians. The question of regularity of the obtained triangulations will be postponed to the end of this section.

Flat fibrations

Let us again start by considering flat fibrations. As we are now dealing with multifibrations, we demand both the elliptic and the K3 fibration to be flat (which also forces the base fibration to be flat). The two-dimensional base fan $\Sigma_{b,K}$ for the K3 fibration is uniquely determined as in section 5.4.1. The fibrations being flat, all rays of $\Sigma_{b,e}$ must pass through meridians. This is required for the fibration π_2 to be flat and already ensured by $\Sigma_{b,K}$ being the fan of the base of a flat fibration.

Now any compatible flat elliptic fibration $\hat{\pi}_e : \Sigma \to \Sigma_{b,e}$ defines triangulations of the sphere segments by lemma 3.1 and Σ being simplicial. The vertices of this triangulation are obviously given by the images of rays over integer points in $\partial^1\nabla$. In addition, Σ must be given by a refinement of the partition of $\partial^1\nabla$ enforced by the K3 fibration, i.e. it must be a refinement of the fan $\tilde{\Sigma}'$ from section 5.4.1. We can now check, whether a flat compatible elliptic fibration exists: We enumerate triangulations of the sphere segments and check, whether a compatible refinement of the initial partition of $\partial^1\nabla$ exists. The latter can again be done by calculating intersection cones and testing, whether they are generated by integer elements of $\partial^1\nabla$.

Finding a compatible triangulation of the sphere is significantly simplified by noting, that the triangulations of the sphere segments are independent: The preimages of the cones over the segments are unions of cones in $\tilde{\Sigma}'$. A triangulation of one sphere segment induces subdivisons of these cones. The intersections of cones projecting to neighboring sphere segments intersect in subcones of the cones projecting to the meridian between the segments. These subcones are in

turn uniquely determined by the points lying on the meridian. Hence, they are independent of the two chosen triangulations.

Due to this uniqueness, one should start the search for a triangulation by checking, whether these subcones are generated by integer elements of $\partial^1\nabla$ (if they are not, no suitable triangulation can exist). For this task, Lemma 5.5 and corollary 5.6 are applicable after restriction to the planes in $N_{b,e}$ defined by the meridians and their preimages under $\hat{\pi}_e$.

The task of enumerating all triangulations of a given sphere segment can be reduced by noting that the images of cones over dimension 1 faces of ∇ must be contained in our triangulation. This yields an initial partition of the sphere segments and only refinements thereof have to be enumerated.

Singularly flat fibrations

Let us now consider cases, in which we were unable to construct base fans and a triangulation compatible with a flat fibration (which means, that none exists). We want to find out whether a singularly flat fibration exists. In accordance with the meaning of the term we demand the elliptic and the K3 fibration to be singularly flat and the base fibration to be flat[53]. Again, our strategy will be to construct the (unique) coarsest base fan for the elliptic fibration giving rise to a singularly flat prefibration. First, we consider the K3 fibration. Assume now, that a fan $\Sigma_{b,K}$ and a compatible triangulation of $\partial^1\nabla$ exist. This triangulation is obviously also compatible with the coarsest possible base fan $\Sigma^c_{b,K}$ leading to a singularly flat fibration, namely the fan defined by the images under $\hat{\pi}_K$ of rays over vertices of ∇. This is a fan due to reflexivity of ∇: If a halfspace of $N_{b,K}$ did not contain the image of a ray over a vertex of ∇ the preimage halfspace would contain no vertex. The latter is a contradiction because 0 is an interior point of ∇.

This coarsest base fan also induces a partition of $\partial^1\nabla$ corresponding to the coarsest fan Σ^c compatible with $\Sigma^c_{b,K}$. Consider the images of the cones in Σ^c under $\hat{\pi}_e$. By construction, each of these images is projected onto exactly one cone of $\Sigma^c_{b,K}$ by $\hat{\pi}_2$. The intersection with the sphere is a spheric polygon, the vertices of which all lie on meridians (either on the same or on two neighboring meridians). A first necessary condition for a singularly flat fibration to exist is given by the meridians themselves, which are subdivided by the points lying on them. The preimage cones in Σ^c must be subdivided to be compatible with this subdivision and the subcones must be generated by integer elements of $\partial^1\nabla$. This can be checked just as before.

The images of cones over one-dimensional borders of the partition of $\partial^1\nabla$ induced by Σ^c induce lines on the sphere between the images of vertices. Lemma 3.1 ensures, that all necessary two-dimensional cones in $\Sigma_{b,e}$ are found in this way. If they patch together to define a (not necessarily simplicial) fan, this is the coarsest possible fan we sought for, and we only have to check whether our

[53] i.e. we demand the fibrations to simultaneously become flat at some boundary of the family of maximally desingularized hypersurfaces

original partition can be refined to be compatible with this fan (we do not worry whether finer fans are possible).

How can the lines fail to define a fan? They certainly define a collection of strongly convex polyhedral cones. The essential condition is that any two cones intersect along a common face of both. This can fail in two ways. There could be (i) points on meridians which fail to emit lines into the neighboring segments (which would mean a ray in the interior of a 2d cone) or there could be (ii) lines which intersect inside the sphere segments (i.e. 2d cones whose intersection is not a face of both). We will first see that (i) cannot happen and then proceed to treat case (ii).

Recall how we constructed the fan Σ^c. The points on meridians are (scaled) images of vertices of ∇ or new vertices of the subdivison of codimension 2 faces of ∇. First consider the second case. If a point is not scaled image of a vertex of ∇, it must be the scaled image of a point n lying on the intersection of a codimension 2 face with a hyperplane in $N_{\mathbb{R}}$ by lemma 5.5 ($n \notin \pi_K^{-1}(0)$). Being a vertex of the subface, it must lie in the interior of an edge of ∇ by corollary 5.6. The two vertices of ∇ connected by the edge must lie on different sides of the hyperplane. As the preimage of the plane in $N_{b,e}$ defined by the meridian (and in particular of the line through the poles) is contained in the hyperplane, the cone over the edge passing through the preimage point must be mapped to the cone over a line crossing the meridian as desired.

We now turn to points whose scaled preimages are vertices of ∇. We will need the following simple consequence of the duality between faces of ∇ and Δ:

Lemma 5.7 *Let v be a vertex of ∇. Any halfspace of N containing v in its boundary contains an edge $\theta^\star \ni v$ of ∇ in its interior.*

Proof: Let θ be the hypersurface of Δ dual to v, i.e. $\forall m \in \theta : \langle m, v\rangle = -1$. Let $m^\circ \in \theta^\circ$ be an interior point. Then $M \supset v^\perp = \mathbb{R}^+\{m - m^\circ; m \in \partial\theta\}$. Let the halfspace be given by $\{n \in N | \langle \hat{m}, n\rangle \leq 0\}$ for some $0 \neq \hat{m} \in v^\perp$. Then $\hat{m} = \mu(m - m^\circ)$ with $\mu \in \mathbb{R}^{>0}, m \in \partial\theta$. Now $m \in \partial\theta \;\Rightarrow\; \exists\theta^\star = \{v + \lambda n; \lambda \geq 0\}$ such that $\forall \tilde{n} \in \theta^\star : \langle m, \tilde{n}\rangle = -1$. As $\lambda > 0 \;\Rightarrow\; \langle m^\circ, v + \lambda n\rangle > -1$, one calculates

$$\langle \hat{m}, v + \lambda n\rangle = \mu\langle m - m^\circ, v + \lambda n\rangle = \mu\left(-1 - \langle m^\circ, v + \lambda n\rangle\right) < 0. \qquad \square$$

In particular, consider the two halfspaces separated by the preimage of the plane on which the meridian lies. Lemma 5.7 ensures the existence of two edges starting in the preimage vertex, which run into the two preimage halfspaces. By projecting the cones over these edges and intersecting their images with the sphere we obtain our desired lines.

All lines on the sphere introduced in the above process are necessary for the resulting fibration to be singularly flat. If some of them intersect between meridians, we thus have to introduce a ray passing through the point of intersection. Since we demand our base fibration to be flat, we also have to introduce a ray in the base fan for the K3 fibration. In section 5.4.1 we already constructed the finest possible fan $\Sigma^f_{b,K}$ giving rise to a singularly flat K3 fibration. If the

point of intersection does not project onto a ray of $\Sigma^f_{b,K}$, we cannot introduce a suitable ray in the base fan for the K3 fibration, and we have ruled out the existence of a singularly flat elliptic K3 fibration. Otherwise, we subdivide $\Sigma_{b,K}$ by introducing the ray and repeat the whole process for this finer fan. Since there are only finetely many rays we can add to the K3 fibration's base fan, we only have to repeat the process finitely many times to either find or exclude the existence of a singularly flat fibration.

In fact, the subdivison does not introduce new lines on the sphere except for the new meridian: Consider a fan $\Sigma_{b,K}$ and the corresponding coarsest compatible fan Σ^c. We now add a new ray ρ_b to $\Sigma_{b,K}$ to obtain the refined fan $\Sigma'_{b,K}$. Let $H^+, H^- \subset N$ be the two halfspaces separated by $\hat{\pi}_K^{-1}(\mathbb{R}\rho_b)$. The coarsest fan compatible with $\Sigma'_{b,K}$ is then given by

$$\begin{aligned} \Sigma'^c &= \{\sigma \mid \sigma \in \Sigma^c \wedge \overset{\circ}{\rho}_b \not\subset \hat{\pi}_K(\overset{\circ}{\sigma})\} \\ &\cup \{\sigma \cap H \mid H \in \{H^+, H^-\} \wedge \sigma \in \Sigma^c \wedge \overset{\circ}{\rho}_b \subset \hat{\pi}_K(\overset{\circ}{\sigma})\}. \end{aligned}$$

Obviously, all two-dimensional cones in Σ'^c are either subsets of two-dimensional cones in Σ^c or project into ρ_b under $\hat{\pi}_K$. Only the latter case induces new lines on the sphere, which lie on the new meridian. In addition, no lines on the sphere vanish as a result of the refinement.

Hence, one can simultaneously introduce new rays for all crossing points and repeat the calculation only once.

Even more general fibrations

As remarked before, checking for all possible base fans is infeasible for more complex examples. We do not win too much by using compatibility with the K3 base fan, since we still would have to consider all K3 fibration base fans given by subsets of the rays in our above $\Sigma_{b,K}$. Since I want to identify as many toric fibration structures as possible, I introduce yet another (weaker) criterion on the fibrations. This allows to identify a large class of fibrations, which are neither flat nor singularly flat.

Namely, I require the K3 fibration to be singularly flat and the base fibration π_2 to be flat. In other words, the elliptic fibration does not have to be singularly flat anymore[54].

For the base fans this criterion means that all rays of $\Sigma_{b,e}$ must project to rays in $\Sigma_{b,K}$ under $\hat{\pi}_2$. Now consider some fixed $\Sigma_{b,K}$ belonging to a singularly flat K3 fibration. Note that the sphere segmentation induced by $\Sigma_{b,K}$ *almost* defines a fan. What is missing is strict convexity of the cones, i.e. it only defines a quasifan. The only thing we have to add to obtain a fan is a path of dividing lines splitting each sphere segment into an upper and lower half. For a given fan $\Sigma_{b,K}$ it is easy to see (c.f. the discussion of regularity below), that any fan $\Sigma_{b,e}$

[54] This is somewhat arbitrary and emerged from some test calculations showing that slightly weaker conditions had much more effect on the speed of calculation than on the number of fibrations found.

giving rise to a flat base fibration π_2 must be a refinement of a fan obtainable in this way.

The points on the meridians through which such a path can possibly run are given by projections of rays over integer points in $\partial^1\nabla$. As before, we denote by Σ^c the coarsest fan compatible with $\Sigma_{b,K}$. The number of points we have to consider is reduced by first checking, whether all cones in Σ^c projecting to the corresponding rays in $\Sigma_{b,K}$ can be split accordingly. Then one can check all lines between points on neighboring meridians for the existence of a compatible refinement of Σ^c. Last, one needs to find a path as described above. This can be done by calculating connection matrices for the points on neighboring meridians, multiplying them in circular order and checking for diagonal elements.

So far, the discussion depends on the choice of $\Sigma_{b,K}$. If one did not demand the K3 fibration to be singularly flat, one would have to enumerate all possibilities. The latter basically consist of enumerating subsets of the set of projections of lines over integer points in $\partial^1\nabla$ and can thus be computationally expensive.

The requirement of a singularly flat K3 fibration reduces the set of possible fans $\Sigma_{b,K}$ in the following ways: Firstly, the set of rays $\Sigma_{b,K}^{(1)} \subset \Sigma_{b,K}$ must be a subset of the rays $\Sigma^{f}{}_{b,K}^{(1)}$ in the finest possible fan $\Sigma_{b,K}^{f}$. Secondly, $\Sigma_{b,K}$ must be a refinement of the coarsest base fan Σ^c. This alone reduces the enumeration to switching on and off elements of $\Sigma^{f}{}_{b,K}^{(1)} - \Sigma^{c}{}_{b,K}^{(1)}$.

In fact, one can do even better than this by introducing cumulative connection matrices for points on the meridians corresponding to neighboring rays in $\Sigma^{c}{}_{b,K}^{(1)}$. An entry of such a matrix is 1 whenever there is a path between the corresponding points for any subdivision of the corresponding two-dimensional cone in $\Sigma^{c(2)}$. The existence of a closed path for any fan $\Sigma_{b,K}$ can then be checked by calculating all the cumulative connection matrices, multiplying them and again checking for diagonal elements[55].

Apart from the obvious reduction of the search space, this approach also has other advantages. The cumulative connection matrices are calculated as follows: One enumerates all possible subdivisions of the cone between the corresponding rays. For all of them, the individual connection matrices are multiplied. The entries of the cumulative matrix are then 1 whenever the corresponding entry in any of these products is nonzero. If after some steps of the enumeration all entries are determined to be 1, further subdivisions do not have to be considered. If, on the other hand, all entries are determined to be 0 after enumerating all subdivisions, further cumulative connection matrices do not have to be calculated. In this case, no closed path can exist.

Both of these cases occur quite frequently.

[55] Usually, one would define the cumulative connection matrices as sums over products for different subdivisions. The entries would then carry information about the number of different paths between points. For the purposes of this paper one only needs to know whether a path exists or not. The definition given here only keeps this information, but allows for the additional optimization described below.

Regularity of the triangulations

For flat and singularly flat K3 fibrations we unfortunately cannot guarantee the enforced partitions to be regular. The situation is different for the more general fibrations discussed in the last paragraph. We will see in a moment that they always allow for compatible regular triangulations.

For flat and singularly flat fibrations we can thereby show that there is a coarser base fan $\Sigma_{b,e}$, which is compatible with a regular triangulation of $\partial^1\nabla$: Any fan Σ_b as constructed for flat and singularly flat fibrations is a refinement of a fan as discussed in the last paragraph. All vertices of the sphere partitions lie on meridians and only segments of the meridians can end in one of the poles. We start at some vertex of the sphere partition induced by $\Sigma_{b,e}$, which is not one of the poles. We then circle around the sphere following lines which cross the sphere segments. If we can choose between multiple lines, we always take the upmost one. Since there are only finitely many lines, we will find a closed path after finitely many steps.

We now consider simplicial base fans $\Sigma_{b,e}$ with rays through the poles and one additional ray through all of the meridians. We will see in a moment, that such a fan is always the fan over the faces of a convex polytope. The inner face normals of it then provide the linear pieces of a stricly convex function on $\Sigma_{b,e}$. As in the two-dimensional case, we use lemma 5.4 and 5.3 to see that at least the coarsest refinement of the fan over faces with codimension ≥ 2 of ∇ compatible with $\Sigma_{b,e}$ is projective.

In order to see that $\Sigma_{b,e}$ is the fan over a not necessarily integer[56] convex polytope, we write $n^+ = (1,0,0)$ and $n^- = (-1,0,0)$ for the poles. We pick vectors $n^{(i)} \in \rho_i$ for all rays $\rho_i \in \Sigma_b$ such that the projections $\pi_2 n^{(i)}$ lie on the unit circle. The $\pi_2 n^{(i)}$ obviously form the vertices of a convex polygon. Now consider the convex hull of all the $n^{(i)}$ together with $r \cdot n^+$ and $r \cdot n^-$, $\mathbb{R} \ni r \gg 1$. For sufficiently large r, these points are vertices of their convex hull and Σ_b is the fan over its faces.

5.5 Search results

After the identification of all virtual elliptic K3 fibrations in the 222,653 polyhedra ∇_{CY} with no more than 10,000 integer points (c.f. section 5.2), all of these fibrations were checked for obstructions as explained above: First, the virtual K3 fibrations were checked for obstructions against giving rise to a K3 fibration as described in section 5.4.1. Depending on the results[57], the full virtual elliptic K3 fibrations were tested as described in section 5.4.2. In addition, the methods of section 4.2 were used to calculate the perturbative gauge algebras of conjectured heterotic duals. Lists containing the results can be found at [Roha].

[56] One can use deformation arguments and density of $\mathbb{Q}$ in $\mathbb{R}$ to find integer polytopes, but for our purposes this is an unnecessary complication.

[57] If e.g. the K3 fibration cannot be flat, checking for flat elliptic K3 fibrations makes no sense.

5.5.1 Het $\leftrightarrow$ Het Dualities

Whenever there are multiple virtual fibration structures within the fourfold polyhedron, one obtains conjectural dualities between different dual heterotic string theories. The perturbative gauge group of one theory is then expected to appear nonperturbatively in the dual theory.

A particularly important case is given by multiple elliptic K3 fibrations sharing the same elliptic fibration[58]. In this setting, one has a single F-theory compactification defined by the elliptic fibration and multiple dual heterotic string compactifications via application of fiberwise duality. Combining such dualities one gets indirect dualities between the different heterotic string compactifications.

Among the tested fibration structures, plenty examples[59] for multiple fibrations of this type were found. Due to their special importance they were collected in separate lists, which may also be found at [Roha].

5.5.2 Counting fibrations

If a reflexive polyhedron ∇_{CY} contains multiple virtual fibrations $\nabla_1^{(f)}, \nabla_2^{(f)} \subset \nabla_{CY}$, the natural question arises which of these should be considered as different. The essential question in this context is, to which degree virtual fibrations related by automorphisms of ∇_{CY} should be identified.

The most simplistic approach is to just fix a representation of ∇_{CY} and to consider virtual fibrations as different, whenever the corresponding lattice subspaces differ[60].

However, there is no special coordinate system – neither for the polyhedra nor the geometric objects they represent. Hence, this does not appear to be a very natural approach. One should rather identify virtual fibrations, which are exchanged by an automorphism of the larger polyhedron ∇_{CY}, i.e. if $\exists g \in GL(N) \cong GL_n(\mathbb{Z}) : g\nabla_{CY} = \nabla_{CY} \ \wedge \ g\nabla_1^{(f)} = \nabla_2^{(f)}$.

The lists at [Roha] exist in two versions corresponding to these two ways of counting elliptic fibrations.

The situation is more involved when counting elliptic K3 fibrations because of the special role of elliptic fibrations shared by multiple elliptic K3 fibrations.

[58] The condition of sharing the same elliptic fibration might be removed by switching to M-theory compactifications, for which the elliptic fibration moduli might be unlocked and smooth transitions between different F-theory limits may be found. Due to the low dimensionality of the corresponding supersymmetric effective theories and the corresponding restrictions the predictive power of such connections is questionable.

[59] There are 92,578 elliptic fibrations (automorphisms modded out, c.f. section 5.5.2) contained in multiple K3 fibrations.

[60] Note that this already mods out automorphisms of the fiber polyhedra.

Consider a multiple virtual fibration

$$\begin{tikzcd}
 & \nabla_{K3}^{(1)} \arrow[dr] & \\
\nabla_{ell} \arrow[ur] \arrow[dr] & & \nabla_{CY} \\
 & \nabla_{K3}^{(2)} \arrow[ur] &
\end{tikzcd}$$

and an automorphism exchanging the K3 fibrations, i.e. $g \in GL(N)$ with $g\nabla_{CY} = \nabla_{CY}$, $g\nabla_{ell} = \nabla_{ell}$ and $g\nabla_{K3}^{(1/2)} = \nabla_{K3}^{(2/1)}$. The two elliptic K3 fibrations are perfectly equivalent structures. Each of them gives rise to a conjectured duality between F-theory and the same heterotic string theory. The automorphism should thus induce an automorphism of the dual heterotic string theory exchanging perturbative with nonperturbative degrees of freedom. For this reason, identification of the two K3 fibrations is not a desirable approach and the lists at [Roha] contain the unreduced numbers of K3 fibrations in both versions.

This often leads to a situation in which some K3 polyhedra are not superpolyhedra of any elliptic polyhedron among a chosen set of representatives. This peculiarity is not easily resolved. Which of the K3 polyhedra are superpolyhedra of elliptic polyhedra among a chosen set of representatives obviously depends on the choice of representatives. Less obviously but worse, the size of the set of superpolyhedra also depends on the choice.

Of course, there is always a lower bound on the size of the set of superpolyhedra, but calculating this bound is computationally expensive and carries no valuable information.

6 Monodromy - two series of examples

In section 4.3 we saw, that in many cases the gauge algebra of a dual heterotic theory cannot be deduced by considering the K3 polyhedron alone.

Among the 1,126,791 virtual elliptic K3 fibrations[61] found, 619,059 K3 polyhedra give rise to nontoric divisors, which might potentially lead to a non-simply-laced gauge group by monodromy. In slightly more than half of these cases (310,396) there actually is a nontrivial action of monodromy on the exceptional fibers leading to non-simply laced gauge algebras. In almost as many cases (279,856) no identifications occur. The discrepancy between the numbers is given by cases, where the (nontoric) section of the elliptic fibration of the K3 fibers patches together to a multisection rather than a section of the total elliptic fibration[62].

[61] The numbers refer to fibrations after identification of elliptic fibrations by automorphisms of ∇.

[62] This happens in 28,807 of the 129,437 examples with nontoric section.

Numbers of fibration structures		
reflexive polyhedra	252,933	
with $\leq 10,000$ integer points	222,653 (88.0 %)	
with virtual elliptic fibration	215,877 (97.0 %)	
with virt. ell. K3 fibration	200,157 (89.9 % / 92.7 %)	
virtual elliptic fibrations	634,827	488,788
virtual K3 fibrations	842,661	
flat K3 fibrations	40,404 (4.8 %)	
sing. flat K3 fibrations	266,092 (31.6 %)	
virtual elliptic K3 fibrations	1,783,067	1,126,791
flat elliptic fibration	19,424 (1.1 %)	16,381 (1.5 %)
sing. flat ell. fibration	67,531 (3.8 %)	53,890 (4.8 %)
more general	523,776 (29.4 %)	395,950 (35.1 %)

Table 3: **Numbers of fibration structures.** *For elliptic and elliptic K3 fibrations the two listed numbers refer to different ways of counting (c.f. section 5.5.2). The numbers for K3 fibrations refer to virtual elliptic K3 fibrations neglecting the elliptic fibration structure. Fibrations fulfilling stronger criteria are not contained in the numbers for weaker criteria, i.e. a flat fibration is not counted as singularly flat.*

Assuming that fourfold polyhedra do not systematically prefer or suppress the occurrence of identifications, this was to be expected[63].

That either possibility should be suppressed was not to be expected, because one can easily construct series of examples for both possibilities. The latter are much simpler than the examples found in my search and yet provide a huge set of examples, if one wishes to concentrate on the question of monodromy. Therefore, I will briefly discuss their construction.

6.1 Monodromy

Examples for cases with complete identification by monodromy are given by the simplest examples one can imagine: trivial fibrations. Of course, only the K3 fibration can be trivial. Otherwise we would obtain no non-abelian gauge group at all. Let Δ_K be a reflexive three-dimensional polyhedron together with a complete triangulation of $\partial\nabla_K$, and Δ_2 a two-dimensional reflexive polyhedron. The direct product of the corresponding toric varieties is given by the product fan which in turn corresponds to the reflexive polyhedron $\Delta = \Delta_K \times \Delta_2$. The dual ∇ is the convex hull of $(\nabla_K, 0)$ and $(0, \nabla_2)$ which obviously has ∇_K as a reflexive subpolyhedron, and we do not need to worry about the virtual toric prefibration being a fibration. It is also clear, that all faces of ∇_K are also faces of ∇ with the same dimension. Hence, no face of ∇_K can be in the

[63] The possibility of independent identifications in the upper and lower half of a polyhedron in some cases increases the probability to find *some* identification.

interior of a codimension 2 face of ∇, and identifications via monodromy occur whenever possible. As a side effect, though, nontoric sections of the K3 fibers' elliptic fibrations patch together to multisections rather than sections of the total elliptic fibration.

6.2 No Monodromy

Examples for this case can easily be constructed using the other (dual) extreme method for constructing reflexive polyhedra. By choosing $\nabla = \nabla_K \times \nabla_2$ all faces of ∇_K lie on faces of ∇ with equal *codimension*. A straightforward but lengthy calculation shows that all of these virtual toric prefibrations define toric fibrations. In addition, one can always find fans such that the K3 fibration becomes flat. Note though that one can only find fans for flat elliptic fibrations in special cases[64].

A Algorithms for calculating lattice bases

In this section, I will give constructive proofs for two lemmata used for the construction of bases for sublattices of $\mathbb{Z}^d$.

Lemma A.1 *Let $B_1 \in \mathbb{Z}^d$ be a primitive vector. Then there exist matrices $B, I \in GL(d, \mathbb{Z})$ with*

$$B = \begin{pmatrix} | & & | \\ B_1 & \cdots & B_d \\ | & & | \end{pmatrix} \qquad \textit{and} \qquad B^{-1} = I = \begin{pmatrix} — & I_1 & — \\ & \vdots & \\ — & I_d & — \end{pmatrix}.$$

Proof: Using the Euclidean algorithm, we can construct a map $C : \mathbb{Z}^d \to \mathbb{Z}^d$ such that $\forall x \in \mathbb{Z}^d : \langle x, C(x) \rangle = \text{g.c.d.}(x)$. First set $I_1 := C(B_1)$. For $n = 2, \ldots, d$ we then construct I_n, B_n such that

$$\forall i \leq n : \langle B_i, I_n \rangle = \langle B_n, I_i \rangle = \delta_{i,n}.$$

To this end choose a primitive $I_n \in \{B_1, \ldots, B_{n-1}\}^\perp$. Set $\tilde{B}_n := C(I_n)$ and finally

$$B_n := \tilde{B}_n - \sum_{i=1}^{n-1} \langle \tilde{B}_n, I_i \rangle B_i.$$

The assertion follows after finitely many steps. □

Lemma A.2 *Let $X := (x_1 \cdots x_n) \in \text{Mat}(\mathbb{Z}; d, n)$ be an integer matrix. Then $\exists \overline{B} := (b_1 \cdots b_d) \in GL(d, \mathbb{Z})$ such that*

$$\text{rk}\,(x_1 \cdots x_r) = s \quad \Rightarrow \quad \langle x_1, \ldots, x_r \rangle_{\mathbb{Q}} = \langle b_1, \ldots, b_s \rangle_{\mathbb{Q}}.$$

[64]Flat fibrations can be obtained if and only if the height (in integer units) of all points in ∇_K above or below the elliptic plane is less than or equal to 1.

Proof: Without loss of generality we can assume the x_i to span $\mathbb{Q}^d$ (otherwise complete with e.g. some vector base of the orthogonal complement). We further assume $\forall i : x_i \neq 0$. We use complete induction on the dimension d. The case $d = 1$ is trivially solved by (1). Assume that the assertion is true for $d-1$. Then set $B_1 := x_1/\text{g.c.d.}(x_1)$ and calculate matrices B, I as in Lemma A.1. Define the matrix $X' \in \text{Mat}(\mathbb{Z}; d-1, n)$ by

$$I\,X =: \begin{pmatrix} \text{---} & \star & \text{---} \\ & X' & \end{pmatrix}.$$

Using the assumption we calculate $\overline{B}' \in GL(d-1, \mathbb{Z})$ and set

$$\overline{B} := B \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \overline{B}' & \\ 0 & & & \end{array} \right).$$

□

B K3 Picard lattices

The following is a short summary of the formulae used to calculate the Picard lattice of a generic toric K3 hypersurface. A derivation and proofs may be found in [Roh04].

Let $\Delta \subset M$ be a reflexive three-dimensional polyhedron, $N \supset \nabla = \Delta^*$. Denote by Z the generic smooth element of the corresponding family of K3 surfaces. The Picard number of Z is then given by

$$\rho(Z) = l(\Delta) - 4 - \sum_{\operatorname{codim}\theta^*=1} l^*(\theta^*) + \sum_{\operatorname{codim}\theta^*=2} l^*(\theta) l^*(\theta^*),$$

where the θ (θ^*) are faces of Δ (∇), $l(X)$ denotes the number of integer points contained in X and $l^*(X)$ the number of integer points in the relative interior. Note that *generic* is a stronger condition than in similar statements for higher dimension: The Picard number is strictly larger for a dense subset of the space of defining polynomials.

For $n \in N \cap \partial^1 \nabla$ denote by D_n the intersection of the corresponding T-Weil divisor with Z, which will be called a toric divisor. D_n is irreducible except for the following situation:

(i) n lies in the interior of an edge θ^* of ∇.

(ii) The length of the dual edge θ of Δ (measured in integer units) is larger than 1.

In this case, D_n splits into $l(\theta) = |m_1 - m_2|$ irreducible components

$$D_n = \sum_{i=1}^{l(\theta)} \tilde{D}_n^{(i)}.$$

Here $l(\theta)$ denotes the integer length of θ, and m_1, m_2 are the two vertices of Δ at the boundary of θ. The irreducible components of D_n are called semitoric [65] divisors.

The nonvanishing intersections between divisors corresponding to different n, n' are:

$$\begin{array}{rcll} D_n \cdot D_{n'} & = & l(\theta) & \text{if } n, n' \text{ are neighbors on a common edge } \theta^\star \\ \tilde{D}_n^{(i)} \cdot D_{n'} & = & 1 & \text{if } n' \text{ is a vertex and } n \text{ is a neighbor in the interior of} \\ & & & \text{a common edge} \\ \tilde{D}_n^{(i)} \cdot \tilde{D}_{n'}^{(j)} & = & \delta_{i,j} & \text{if } n, n' \text{ are neighbors in the interior of a common edge} \end{array}$$

The self-intersections are determined by the general rule

$$D_n \cdot D_n = \sum_{n'} \langle m, n' \rangle D_n \cdot D_{n'},$$

where $m \in M$ such that $\langle m, n \rangle = -1$ (e.g. a normal of some face on which n lies) and the sum ranges over all neighbors n' of n on common edges.

In particular, for n in the interior of some edge of ∇, one obtains

$$\begin{array}{rcl} D_n \cdot D_n & = & -2l(\theta) \\ \tilde{D}_n^{(i)} \cdot \tilde{D}_n^{(i)} & = & -2. \end{array}$$

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References

[AKM00] P. S. Aspinwall, S. Katz und D. R. Morrison, *Lie groups, Calabi-Yau threefolds, and F-theory*, Adv. Theor. Math. Phys. **4**, 95–126 (2000), hep-th/0002012.

[AKMS97] A. Avram, M. Kreuzer, M. Mandelberg und H. Skarke, *Searching for K3 Fibrations*, Nucl. Phys. **B494**, 567–589 (1997), hep-th/9610154.

[65] in contrast to the divisors not equivalent to divisors with support in $X_\Delta - (\mathbb{C}^\star)^3$, which emerge at a dense subset of the parameter space

[AL96] P. S. Aspinwall und J. Louis, *On the Ubiquity of K3 Fibrations in String Duality*, Phys. Lett. **B369**, 233–242 (1996), hep-th/9510234.

[Aud91] M. Audin, The Topology of Torus Actions on Symplectic Manifolds, in *Progress in Math.*, volume 93, Birkhäuser, 1991.

[Bat93] V. V. Batyrev, Quantum cohomology rings of toric manifolds, in *Journées de Géométrie Algébrique d'Orsay (Juillet 1992)*, volume Astérisque 218, pages 9–34, Société Mathématique de France, 1993.

[Bat94] V. V. Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, J. Alg. Geom. **3**, 493–535 (1994), alg-geom/9310003.

[BB97] V. V. Batyrev und L. A. Borisov, Dual cones and mirror symmetry for generalized Calabi-Yau manifolds, in *Mirror Symmetry, II*, edited by B. Greene und S.-T. Yau, pages 71–86, Providence, RI, 1997, Amer. Math. Soc., alg-geom/9402002.

[BCOG00] V. Braun, P. Candelas, X. D. L. Ossa und A. Grassi, *Toric Calabi-Yau fourfolds, duality between $N = 1$ theories and divisors that contribute to the superpotential*, (2000), hep-th/0001208.

[BKMT99] P. Berglund, A. Klemm, P. Mayr und S. Theisen, *On Type IIB Vacua With Varying Coupling Constant*, Nucl. Phys. **B558**, 178–204 (1999), hep-th/9805189.

[Bor86] C. Borcea, *Diffeomorphisms of a K3 surface*, Math. Ann. **275**, 1–4 (1986).

[BPS99] M. Bershadsky, T. Pantev und V. Sadov, *F-Theory with Quantized Fluxes*, Adv. Theor. Math. Phys. **3**, 727–773 (1999), hep-th/9805056.

[BSW97] R. Blumenhagen, R. Schimmrigk und A. Wisskirchen, *(0,2) mirror symmetrie*, Nucl. Phys. **B486**, 598 (1997), hep-th/9609167.

[CF98] P. Candelas und A. Font, *Duality between the webs of heterotic and Type II vacua*, Nucl. Phys. **B511**, 295–325 (1998), hep-th/9603170.

[Cox95] D. A. Cox, *The homogenous coordinate ring of a toric variety*, J. Algebraic Geom. **4**, 17–50 (1995).

[Cox97] D. A. Cox, Recent Developments in Toric Geometry, in *Algebraic geometry — Santa Cruz 1995*, edited by J. Kollar et al., pages 389–436, Providence, RI, 1997, Amer. Math. Soc., alg-geom/9606016.

[Don90] S. Donaldson, *Polynomial Invariants For Smooth Four-Manifolds*, Topology **29**, 257–315 (1990).

[Ful93] W. Fulton, *Introduction to Toric Geometry*, Springer-Verlag, Berlin Heidelberg New York, 1993.

[GSO76] F. Gliozzi, J. Scherk und D. Olive, *Supergravity and the spinor dual model*, Phys. Lett. **65B** (1976).

[GSO77] F. Gliozzi, J. Scherk und D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl. Phys. **B122** (1977).

[KLRY98] A. Klemm, B. Lian, S.-S. Roan und S.-T. Yau, *Calabi-Yau fourfolds for M- and F-Theory compactifications*, Nucl. Phys. **B518**, 515–574 (1998), hep-th/9701023.

[KS94] A. Klemm und R. Schimmrigk, *Landau-Ginzburg string vacua*, Nucl. Phys. **B411**, 559–583 (1994), hep-th/9204060.

[KS98a] M. Kreuzer und H. Skarke, *Calabi-Yau 4-folds and toric fibrations*, J. Geom. Phys. **26**, 272–290 (1998), hep-th/9701175.

[KS98b] M. Kreuzer und H. Skarke, *Classification of Reflexive Polyhedra in Three Dimensions*, Adv. Theor. Math. Phys. **2**, 847–864 (1998), hep-th/9805190.

[KS00] M. Kreuzer und H. Skarke, *Complete classification of reflexive polyhedra in four dimensions*, (2000), hep-th/0002240.

[LSW99] M. Lynker, R. Schimmrigk und A. Wisskirchen, *Landau-Ginzburg-Vacua of string-, M- and F-theory at c=12*, Nucl. Phys. **B550**, 123–150 (1999), hep-th/9812195.

[Mat85] T. Matumoto, On Diffeomorphisms of a K3 surface, in *Algebraic and Topological theories — to the memory of Dr. Takehiko Miyaka*, edited by M. Nagata et al., pages 616–621, Kinokuniya, Tokyo, 1985.

[Mus94] I. Musson, *Differential operators on toric varieties*, J. Pure and Appl. Alg. **95**, 303–315 (1994).

[Nik80] V. V. Nikulin, *Finite automorphism groups of Kaehler K3 surfaces*, Trans. Mosc. Math. Soc. **38**, 71–135 (1980).

[PS97] E. Perevalov und H. Skarke, *Enhanced Gauge Symmetry in Type II and F-Theory Compactifications: Dynkin Diagrams From Polyhedra*, Nucl. Phys. **B505**, 679–700 (1997), hep-th/9704129.

[Roha] F. Rohsiepe, List of hypersurfaces in weighted projective spaces, reflexive Newton polyhedra and fibration structures, http://www.th.physik.uni-bonn.de/People/rohsiepe.

[Rohb] F. Rohsiepe, Lists of elliptic toric K3 surfaces, sections and gauge algebras, http://www.th.physik.uni-bonn.de/People/rohsiepe.

[Roh04] F. Rohsiepe, *Lattice polarized toric K3 surfaces*, (2004), hep-th/0409290.